GREEDY ALGORITHMS

MATH CIRCLE (HS1) 11/17/2013

Introduction

1) Let x/y be a reduced fraction. If we write

$$\frac{x}{y} = \frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_k}$$

we call the right hand side an *Egyptian Fraction* with length k. Write each of the following as an Egyptian fraction:

$$\text{a) } \frac{2}{3} \text{ b) } \frac{3}{4} \text{ c) } \frac{3}{5} \text{ d) } \frac{4}{7} \text{ e) } \frac{11}{12} \text{ f) } \frac{4}{5} \text{ g) } \frac{5}{91} \text{ h) } \frac{9}{11}$$

Hints: For g), use multiples of 7. For h), this is fairly difficult.

2) a) Suppose you have the following choices for activities (all times AM): Swimming, 10:15-11:15; Basketball, 9:30-10:45; Homework, 11-11:45; Reading 8:45-9:15; TV 9-9:45; Nap, 11-12. Plan a morning that has the most activities possible.

b) Suppose you have activities $p, q, r, s, t, u, v, w, x, y, z$ with start and end times

$(1, 4), (3, 5), (0, 6), (5, 7), (3, 8), (5, 9), (6, 10), (8, 11), (8, 12), (2, 13), (12, 14)$.

Find a maximal set of activities that do not overlap.

3) For the graphs on the next page: Find the shortest path (by weight) from each vertex to the starred vertex.

Some problems adapted from previous Math Circle handouts made by Clint Givens.

Egyptian Fractions

1) Suppose $y = q \cdot x + r$ with $q \geq 0$ and $0 < r < x$.

a) Show $\frac{x}{y} = \frac{1}{q+1} + \frac{x-r}{y(q+1)}$. (Equivalently: $\frac{1}{\lceil y/x \rceil} + \frac{(-y) \bmod x}{y \cdot \lceil y/x \rceil}$.)

b) Define a greedy algorithm for computing Egyptian Fractions. Why does it make sense to call this algorithm greedy?

c) Does this algorithm give efficient answers to the fractions in Intro. 1)?

2) a) Prove that x/y has an Egyptian Fraction of length $\leq x$.

b) Find y such that any Egyptian Fraction for $3/y$ has length 3.

(Note: It is conjectured that $4/y$ has an Egyptian Fraction of length ≤ 3 for all y.)

c) Challenge: For what y does $3/y$ have an Egyptian Fraction of length 3?

Activity Selection

1) Show that the following greedy algorithm is NOT optimal.

Greedy Algorithm: At each stage, take the shortest event that doesn't overlap with the events already chosen.

2) Come up with a greedy algorithm that gives optimal choices for the activites in Intro. 2), as well as the sets below:

i) $(1, 5), (0, 6), (3, 8), (7, 16), (6, 12), (5, 14), (10, 19), (12, 18)$

ii) $(1, 2), (0, 4), (6, 8), (7, 9), (3, 6), (8, 11), (2, 5), (5, 9), (9, 12), (10, 14), (11, 13)$.

3) Challenge: Show by induction that the greedy algorithm you described in 2) is always optimal.

Shortest Paths

1) Explain why a greedy algorithm will not work well for computing shortest paths to a starred vertex. Examples will help!

2) a) Come up with an algorithm/systematic way for computing shortest paths to a starred vertex.

b) Use your answer in a) to compute the shortest paths to the starred vertices in Primland and Kruskalia.

Games

1) Suppose you have 3 stacks of 7 stones. Show that the first player can win the following games:

a) Take turns taking between 1-4 stones, all from the same stack. Whoever gets the most stones wins.

b) Same as a), except if you take all remaining stones in a stack, you get another turn.

2) Show that, in general, greedy algorithms are not very good for playing games. For example, think about checkers/chess, dots, monopoly, etc.