COMPLEX NUMBERS AND GEOMETRY

MATH CIRCLE (HS1) 11/10/2013

We can visualize complex numbers by thinking of the point (a, b) in the plane as the complex number $z = a + bi$.

Note that under this interpretation $|z|$ is simply the distance from (a, b) to the origin.

Transformations

1) Suppose we have complex numbers $z_1 = 2$, $z_2 = 3i$, $z_3 = -2+i$, $z_4 = 3-2i$, $z_5 = -1-i$.

For each of the mappings below, plot $z_1, \dots, z_5$ before and after applying the mapping:

a) $z \mapsto 2z$

b) $z \mapsto iz$

c) $z \mapsto -iz$

d) $z \mapsto i(z + 1) - 1$

e) $z \mapsto i(iz + 1)$

2) For each of the following, give a written description of the transformation described:

a) $z \mapsto -z$

b) $z \mapsto -iz$

c) $z \mapsto \frac{1}{1+i}(z - i)$

d) $z \mapsto 2z + 1$

e) $z \mapsto \overline{z}$

3) Come up with a complex mapping that corresponds to the following transformations:

a) Rotation by 90° counterclockwise about the origin.

b) Rotation by 180° clockwise about the origin.

c) Reflection about the horizontal axis.

d) Translation 3 units up and 2 to the right.

e) Reflection about the vertical axis. Hint: Think about combining transformations!

f) Rotation by 90° clockwise about the point $-2, 1$.

g) Reflection over a line through the origin at a 45° angle with the horizonal axis.

Some problems adapted from previous Math Circle handouts.

Challenge: Applications to Geometry

We view points in the plane as complex numbers as above.

1) a) Show that z_1, z_2, z_3 are collinear if and only if $(z_1 - z_2)/(z_2 - z_3)$ is a real number.

b) Show that the lines through z_1, z_2 and z_3, z_4 are perpendicular if and only if $(z_1 - z_2)/(z_3 - z_4)$ is imaginary.

2) Let w be a complex number, and $r \in \mathbb{R}$.

a) Convince yourself that $|z - w| = r$ is the equation of a circle.

b) Find $s \in \mathbb{R}$ so that $z\overline{z} - (z\overline{w} + \overline{z}w) + s = 0$ has the same graph as$|z - w| = r$.

3) a) Explain the transformation $z \mapsto 1/z$ in your own words.

b) Prove that under this transformation, circles are mapped to circles.