

HILBERT'S HOTELS (AN INTRODUCTION TO SET THEORY)

LAMC INTERMEDIATE/ADVANCED GROUPS - 2/09/14

(1) Write down the definitions of the following, and give an example when possible.

(a) Two sets **Have the same number of elements:**

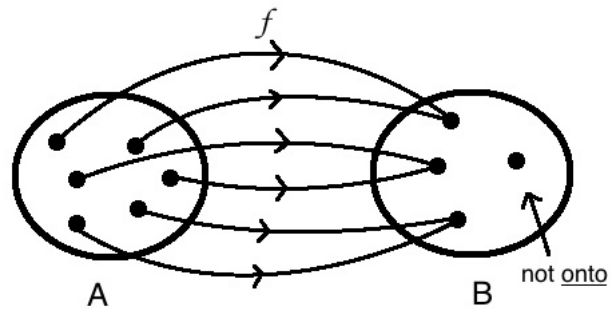
(b) A function between two sets is **One-to-One:**

(c) A function between two sets is **Onto:**

(d) A function between two sets is a **Bijection:**

(e) Give a more “grown-up” definition of two sets “having the same number of elements.”

(2) Here is a picture of a function that is not one-to-one or onto.



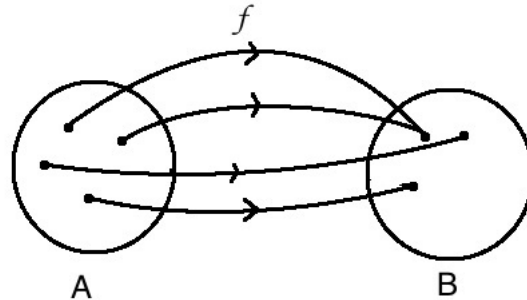
(a) Draw your own picture of a function that is onto but not one-to-one.

(b) Draw your own picture of a function that is one-to-one but not onto.

(c) Draw your own picture of a function that is one-to-one and onto.

(3) Determine which of the following functions are onto, one-to-one, and bijections.

(a) The function f represented pictorially as:



(b) The function $f(n) = n$ from \mathbb{N} to \mathbb{N} .

(c) The function $f(n) = n + 5$ from \mathbb{N} to \mathbb{N} .

(d) The function $f(n) = n \pmod{2}$ from \mathbb{N} to \mathbb{N} . What is the image of this function?

(4) One-to-One functions, Onto functions, and Bijections.

(a) Can you come up with an onto function from the set $\{a, b, c, \dots, z\}$ to the set $\{1, 2, 3, \dots, 25\}$?

(b) What about an onto function from $\{1, 2, 3, \dots, 25\}$ to $\{a, b, c, \dots, z\}$?

(c) Is there a bijection from $\{1, 2, 3, 4, \dots, 10\}$ to $\{2, 4, 6, 8, \dots, 20\}$? If so, give an explicit formula for the function.

(d) Is there a bijection from $\{2, 4, 6, 8, \dots, 20\}$ to $\{1, 2, 3, 4, \dots, 10\}$?

(e) Is there a one-to-one function from the set of students in the room to the set of chairs in the room?

(5) In this problem A is a set with n elements and B is a set with m elements (m and n are natural numbers).

(a) Suppose there is a one-to-one function $f : A \rightarrow B$. What (if anything) can we say about the relationship between m and n ?

(b) Suppose there is an onto function $f : A \rightarrow B$. What (if anything) can we say about the relationship between m and n ?

(c) Suppose there is a bijection $f : A \rightarrow B$. What (if anything) can we say about the relationship between m and n ?

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- (d) Suppose $n \leq m$. Show that we can create an onto function $g : B \rightarrow A$.
- (e) Suppose $n \geq m$. Show that we can create a one-to-one function $g : B \rightarrow A$.
- (f) Suppose $n = m$. Show that we can create a bijection $g : A \rightarrow B$.
- (g) (Baby Cantor-Schroeder-Bernstein) Show that if we have a one-to-one function $f : A \rightarrow B$ and a one-to-one function $g : B \rightarrow A$ then we have a bijection $h : A \rightarrow B$. **(Draw a picture!)**

(6) Using the definitions above, try to come up with your own definition of the word “**infinite.**”

(7) Using your definition, prove rigorously that:

(a) \mathbb{N} is infinite.

(b) The set of Math Circle instructors is not infinite.

(c) The set \mathbb{Z} is infinite.

HILBERT'S HOTEL

Hilbert's Hotel is the most popular hotel in the galaxy, located just light-years away from the edge of the Milky Way. One day in the not-so-distant future a traveller from Earth arrived at the hotel:

"Are there any rooms vacant?"

"At the moment all rooms are occupied by other guests." Replied the hotel clerk.

"Is there anything you can do?" replied the traveller, "I really need a room for tonight."

The clerk replied, "Okay, here is your room key. Room number one."

Explanation.

- (1) An explanation of this is as follows: The hotel has infinitely many rooms $1, 2, 3, \dots$. When the traveller arrives, the hotel clerk asks the guest in room 1 to move into room 2, the guest in room 2 to move into room 3, etc. Explain this process using a one-to-one function.

- (2) Explain how the Hilbert Hotel can accommodate the following:

- (a) All of the Math Circle instructors need rooms, but the hotel is full.

- (b) New people, called $\{p_1, p_2, p_3, \dots\}$, need rooms, but the hotel is full.

(3) Can you find bijections between the following sets? (**Hint:** Once again, it will be useful to draw diagrams)

(a) $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $2\mathbb{N} = \{0, 2, 4, 6, \dots\}$ (even numbers).

(b) \mathbb{N} and $2\mathbb{N} + 1 = \{1, 3, 5, \dots\}$ (odd numbers).

(c) \mathbb{N} and $\{A, 0, 1, 2, 3, \dots\}$

(d) \mathbb{N} and $\mathbb{N} + 5 = \{5, 6, 7, \dots\}$

(e) \mathbb{N} and $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$ (**Hint:** It is easy to split up \mathbb{Z} into two groups, positive and negative. Can you find a similar grouping for \mathbb{N} ? Once you find the grouping, come up with a bijection.)

HOMEWORK

- (1) Show that the set of evens and the set of odds have the same size.
- (2) There is a sporting event near the Hilbert Hotel and all the teams need a place to stay. The hotel was originally empty, but there are infinitely many teams, and each team has infinitely many players. Can the Hilbert Hotel accommodate all the teams?
- (Hint:** You'll have to break up the hotel into infinitely many groups of infinite size. Note that there are infinitely many primes, and for example, the sets $\{2, 2^2, 2^3, \dots\}$ and $\{3, 3^2, 3^3, \dots\}$ are disjoint.)