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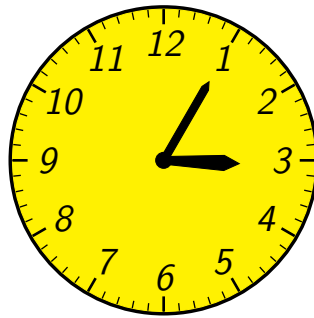
Warm-up

Variations of the following two problems were communicated to me by one of our students, Arul Kolla. They are fun, plus they originate from the 2013 Math Kangaroo test for grades 5 and 6, an extra bonus for us. (Thanks, Arul!)

Problem 1 *There were 2014 inhabitants on an island. Some of them were knights and the others were liars. The knights always tell the truth while the liars always lie. Every day, one of the inhabitants said, "After my departure, the number of knights on the island will be equal to the number of liars," and then left the island. After 2014 days there was nobody left on the island. How many liars were there initially?*

Problem 2 *The program “ChangeSum” is given a list of three numbers, 20, 1, and 4. The program creates a new list by replacing each given number by the sum of the other two. This way, the program replaces 20 by 5, 1 by 24, and 4 by 21. The program runs 2013 times using the output of the previous cycle as the input of the next one and stores all the 2014 triples of numbers in the memory. What is the maximal difference between two elements in a triple among all the triples?*

Problem 3 *What is the angle between the hour and the minute hand of a clock at 3:05?*



Problem 4 *Write down the definition of a rational number.*

Problem 5 *Prove that $\sqrt[3]{19}$ is not rational.*

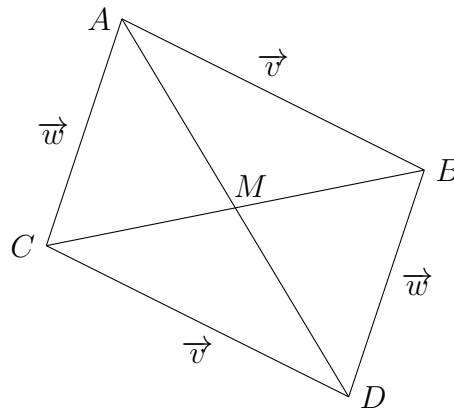
Problem 6 *Find $\lfloor \sqrt[3]{19} \rfloor$ and $\lceil \sqrt[3]{19} \rceil$.*

Back to vectors

The following are Example 1 and Problems 7 – 17 from our 1/19/2014 class.¹ We did not solve them back then. Let's solve them now.

Example 1 *Prove that diagonals of a parallelogram in the Euclidean plane split each other in halves.*

Consider the below parallelogram $ABCD$. Let $\overrightarrow{AB} = \vec{v}$ and $\overrightarrow{AC} = \vec{w}$. Let M be the midpoint of the diagonal AD . To prove the statement, we need to show that the diagonal CB also passes through M and that $|CM| = |MB|$.



According to the definition of vector addition (a.k.a. the parallelogram rule), $\overrightarrow{AD} = \vec{v} + \vec{w}$. Hence, $\overrightarrow{AM} = .5(\vec{v} + \vec{w})$.

According to the definitions of an opposite vector and of vector addition, $\overrightarrow{CB} = \vec{v} - \vec{w}$. Hence, the vector that originates

¹ <http://www.math.ucla.edu/~radko/circles/lib/data/Handout-625-740.pdf>

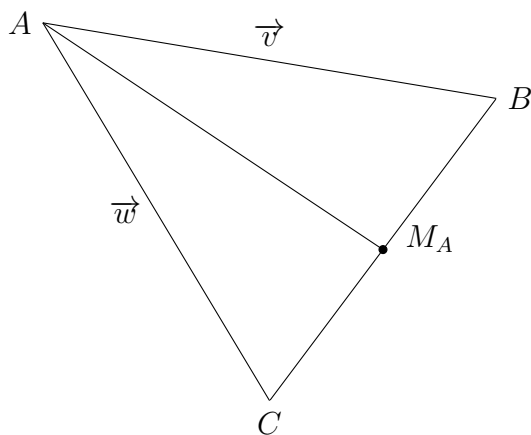
at C and terminates at the midpoint of the diagonal CB is $.5(\vec{v} - \vec{w})$. Let us add up \vec{w} and this vector.

$$\vec{w} + .5(\vec{v} - \vec{w}) = .5(\vec{v} + \vec{w}) = \overrightarrow{AM}$$

In other words, if we first walk along the vector \overrightarrow{AC} and then continue along the vector that originates at C and terminates at the midpoint of the diagonal CB , we end up at M , the midpoint of the diagonal AD . Therefore, the midpoints of the diagonals coincide. *Q.E.D.*

Following Archimedes of Syracuse, we have used geometry of weights to prove that all the three medians of a triangle intersect at one point that splits each median in the ratio 2 : 1 counting from the vertex.² In the following sequence of problems, we will re-discover this wonderful fact using vector algebra.

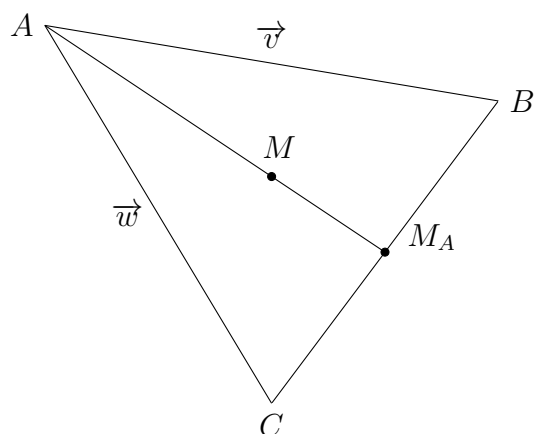
Problem 7 Consider the triangle ABC below. Let $\overrightarrow{AB} = \vec{v}$ and $\overrightarrow{AC} = \vec{w}$. Let M_A be the midpoint of the side BC .



²<http://www.math.ucla.edu/~radko/circles/lib/data/Handout-510-612.pdf>

Use the parallelogram rule to find the numbers a and b such that $\overrightarrow{AM_A} = a\vec{v} + b\vec{w}$. In other words, express $\overrightarrow{AM_A}$ as a linear combination of \vec{v} and \vec{w} .

Problem 8 Let M be a point of the median AM_A such that $|AM| = 2|MM_A|$.

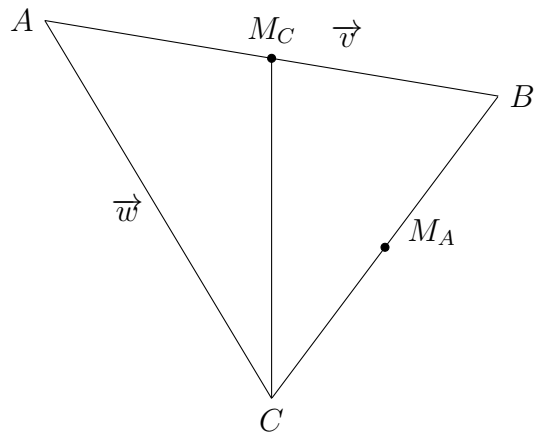


Express \overrightarrow{AM} as a linear combination of \vec{v} and \vec{w} . Simplify the coefficients of the expression, the numbers a and b such that $\overrightarrow{AM} = a\vec{v} + b\vec{w}$, as much as possible.

Let M_C (on the picture below) be the midpoint of the side AB . We need to show that

1. the line CM_C passes through M ; and
2. $|CM| = 2|MM_C|$.

Problem 9 Express $\overrightarrow{CM_C}$ as a linear combination of \vec{v} and \vec{w} .



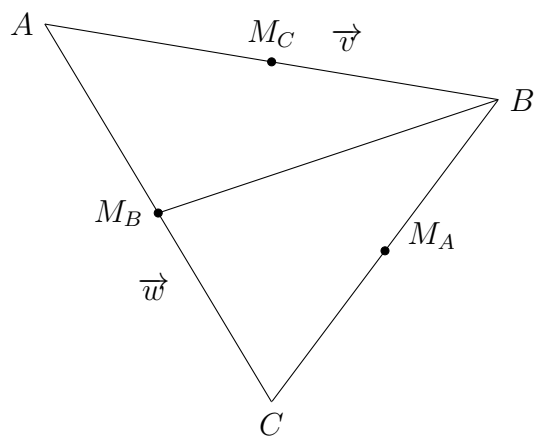
Problem 10 Represent the vector

$$\vec{w} + \frac{2}{3}\overrightarrow{CM_C}$$

as a linear combination of \vec{v} and \vec{w} . Compare the result to \overrightarrow{AM} .

Let M_B be the midpoint of the side AC .

Problem 11 Express $\overrightarrow{BM_B}$ as a linear combination of \vec{v} and \vec{w} .



Problem 12 Represent the vector

$$\vec{v} + \frac{2}{3}\overrightarrow{BM_B}$$

as a linear combination of \vec{v} and \vec{w} . Compare the result to $\overrightarrow{AM_A}$.

We have proven the median theorem. However, a double-check never hurts.

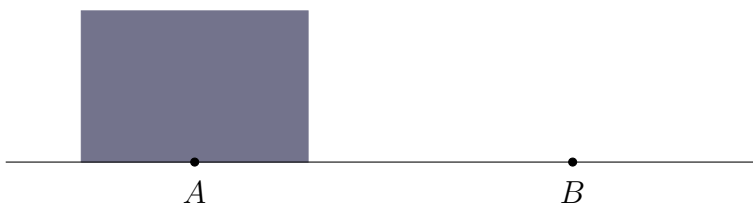
Problem 13 *Represent the vector*

$$\frac{1}{2}\vec{w} + \frac{1}{3}\overrightarrow{M_B B}$$

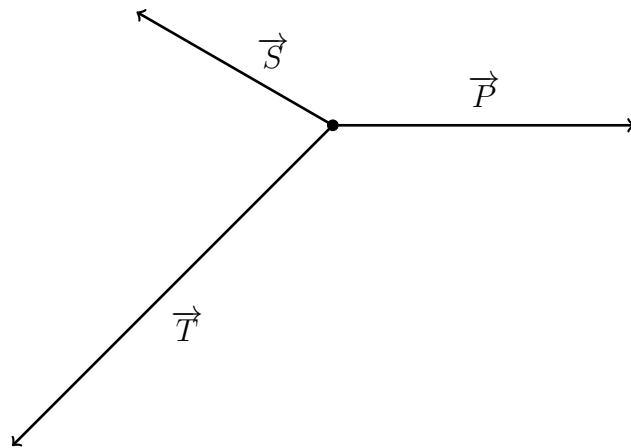
as a linear combination of \vec{v} and \vec{w} . Compare the result to \overrightarrow{AM} .

As the following problem shows, vectors are a great tool not only in mathematics, but also in physics.

Problem 14 *You need to slide a heavy box over the floor from point A to point B. The box is about twice as short as you are. Which way is easier, to push or to pull? Why?*



Problem 15 *The dot on the picture below represents a spaceship. (Compared to the vastness of space, spaceships do look like dots.) There are three forces acting on the ship. \vec{T} is the thrust of the ship's engine. \vec{P} is the gravitational pull of the neighbouring planet. \vec{S} is the gravitational pull of the planet's home star. You are the captain. Use a compass and a ruler to figure out where the resulting force would steer the ship.*



Next time, we will use vectors to formulate and study the laws of Newtonian mechanics, and more. In the meantime...

Problem 16 *Sum up all the integers from one to a thousand.*

Problem 17 *Do the same for the octals. Write down the answer in both the octal and the decimal form. Indicate the base by a subindex.*