

# Introduction to Cardinality

LAMC Intermediate/Advanced Groups

02/09/14

1. Write down the definitions of the following, and give an example when possible:

(a) Have the same number of elements:

(b) : One-to-One (Injection):

(c) Onto (Surjection):

(d) Bijection

(e) Give a more "grown-up" definition of two sets "having the same number of elements."

2. Injections, Surjections, and Bijections.

(a) Can you come up with a surjection from the set  $\{a, b, c, \dots, z\}$  to the set  $\{1, 2, 3, \dots, 25\}$ ?

(b) What about a surjection from  $\{1, 2, 3, \dots, 25\}$  to  $\{a, b, c, \dots, z\}$ ?

(c) Is there a bijection from  $\{1, 2, 3, 4, \dots, 10\}$  to  $\{2, 4, 6, 8, \dots, 20\}$ ? If so, give an explicit formula for the function.

(d) Is there a bijection from  $\{2, 4, 6, 8, \dots, 20\}$  to  $\{1, 2, 3, 4, \dots, 10\}$ ?

3. Let  $A, B$ , and  $C$  be sets,  $f : A \rightarrow B$  and  $g : B \rightarrow C$  functions. Decide if the following are true or false, and then justify your answer.

(a) If  $f$  and  $g$  are surjections, then the function  $g(f(a)) : A \rightarrow C$  is a surjection.

(b) If  $f$  and  $g$  are injections then the function  $g(f(a)) : A \rightarrow C$  is an injection.

(c) If  $f$  is a surjection and  $g$  is an injection, then the function  $g(f(a)) : A \rightarrow C$  is a bijection.

4. Using our definitions, try to come up with your own definition of "infinite"

5. Hilbert's Hotel is the most popular hotel in the galaxy, with infinitely many rooms, located just light-years away from the edge of the Milky Way. One day in the not-so-distant future, a traveller from Earth arrived at the hotel:

"Are there any rooms vacant?"

"At the moment all rooms are occupied by other guests," replied the hotel clerk.

"Is there anything you can do?" replied the traveler, "I really need a room for tonight."

The clerk replied, "Okay, here is your room key. Room number one."

- (a) We can explain that interaction as follows: the hotel has infinitely many rooms numbered  $1, 2, 3, \dots$ , and the guest in room 1 moves to room 2, the guest in room 2 moves to room 3, etc. Explain this process using an injection (Note: we can think of this as  $1 + \infty = \infty$ )

- (b) Come up with a story to explain  $17 + \infty = \infty$

- (c) Come up with a story to explain  $\infty + \infty = \infty$

6. Can you find bijections between the following sets?

(a)  $\mathbb{N} = \{0, 1, 2, 3, \dots\}$  and  $2\mathbb{N} = \{0, 2, 4, 6, \dots\}$

(b)  $\mathbb{N}$  and  $\{\text{Math Circle}, 0, 1, 2, 3, \dots\}$

(c)  $\mathbb{N}$  and  $5 + \mathbb{N} = \{5, 6, 7, \dots\}$

(d)  $\mathbb{N}$  and  $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$  (**Hint:** It is easy to split up  $\mathbb{Z}$  into two subsets, positive and negative. Can you find a similar grouping for  $\mathbb{N}$ ?)

7. Now, let's try a slightly harder example, showing that there is a bijection from  $\mathbb{N}$  to  $\mathbb{N} \times \mathbb{N}$ , the set of ordered pairs of natural numbers.

(a) How many elements  $(a, b) \in \mathbb{N}$  have  $a + b \leq 5$ ? How many have  $a + b \leq 10$ ?

(b) Represent  $\mathbb{N} \times \mathbb{N}$  as a 2-dimensional coordinate grid.

(c) Show that there is a bijection from  $\mathbb{N}$  to  $\mathbb{N} \times \mathbb{N}$

8. Is there a bijection between the following sets?

(a)  $\mathbb{N}$  and  $\mathbb{Q}^+$ , the set of positive rational numbers

(b)  $\mathbb{Z}$  and  $\mathbb{Q}$

(c)  $\mathbb{N}$  and  $\mathbb{Q}$

9. Now let's think about whether it's possible to come up with a bijection from  $\mathbb{N}$  to  $\mathbb{R}$ . Suppose we have a one-to-one function from the natural numbers to real numbers between 0 and 1,  $f : \mathbb{N} \rightarrow (0, 1)$ . Since any real number between 0 and 1 can be written uniquely as a decimal without trailing 9's, we can say that the decimal representation of  $f(n)$  is  $.f(n)_1f(n)_2f(n)_3\dots$ . Now, let's define a number  $x$  as  $x = .(f(1)_1 \pm 5)(f(2)_2 \pm 5)(f(3)_3 \pm 5)\dots$ , where for each place, we choose between add or subtract by which option keeps the value between 0 and 9.

(a) Prove that there is no  $n \in \mathbb{N}$  such that  $f(n) = x$ .

(b) What does this fact say about the possibility of a bijection between  $\mathbb{N}$  and  $(0, 1)$ ?

(c) Is there a bijection between  $(0, 1)$  and  $\mathbb{R}$ ?

(d) Is there a bijection between  $\mathbb{N}$  and  $\mathbb{R}$ ?