Inequalities: What's Greater?

Los Angeles Math Circle | Advanced Group | 01.26.2014

In this unit we will learn the skill of fast and precise estimation by solving problems with specific numerical data.

1. Warm-up/Review. Simplify the following. If it is practical to give your answer as a number, do so. Otherwise, leave it in terms of a power or product of powers.

   (a) 3^4

   (b) $(2^2)^{15}$

   (c) $(3^5) \cdot (3^2)^3$

   (d) $2 \cdot 2 \cdot (2^4)^2$

   (e) $(a \cdot b^x)^y \cdot (a^y) \cdot \frac{1}{b^3}$

2. Let's see which number is greater: 31^{11} or 17^{14}. Clearly, it would be *possible* to calculate both numbers "manually" with a calculator that gave you enough decimal places. But this way of dealing with the problem, even if we got lucky here, will not help us solve more complex problems. Let us try another way. I'll guide you.

   (a) We need a way to relate these numbers. We can do this either by using the bases or the exponents. In this case, note that the bases are both very close to powers of 2. We'll use this to relate them. What are these powers of 2?

(b) Start with 31^{11}. Using an inequality, relate it to 32^{11}, which is $(2^5)^{11}$.

(c) Now, observe that $(2^5)^{11} = 2^{55}$, which is less than 2^{56}. Write this in terms of 2^4. Can you explain why we have chosen 2^4 as our base?

(Note that we are going in the direction of attempting to show that $31^{11} < 17^{14}$.)

(d) Can you show that your answer to part (b) is smaller than 17^{14}?

(e) Summarize your answers to parts (a)-(d) using only numbers and math symbols. For example, if we were trying to show that 3^4 is smaller than 244, you might write $3^4 < 3^5 = 243 < 244$.

3. Which number is greater? In each case, consider whether it would be more logical to relate the exponents or the bases.

(a) 2^{300}or 3^{200}?

(b) 2^{42} or 3^{28}?

(c) 5^{44} or 4^{53}?

4. Prove that $2^{100} + 3^{100} < 4^{100}$.

5. Which number is greater: 7^{92} or 8^{91}?

6. Which number is greater: $1234567 \cdot 1234569$ or 1234568^2?

7. We are given the two fractions

$$\frac{10...01}{10...001} \text{ and } \frac{10...01}{10...001}.$$

In each fraction the number in the numerator has one zero more than the one in the denominator. If the numerator in the left fraction has 2013 zeroes, and the numerator in the right fraction has 2014 zeroes, which of them is greater?

8. Under what condition is $\frac{x}{y} > \frac{x+1}{y+1}$?
(Hint: Subtract them.)

9. Which number is greater: $\frac{1234567}{7654321}$ or $\frac{1234568}{7654322}$?

10. Which number is greater: 100^{100} or $50^{50} \cdot 150^{50}$?
(Hint: $100^2 > 100 \cdot 50$)

11. Prove that

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - ... - \frac{1}{99} + \frac{1}{100} > \frac{1}{5}.$$

12. If you apply the factorial function 99 times to the number 100, then you get the number A. If you apply the factorial function 100 times to the number 99, you get the number B. Which of them is greater?

13. Find all powers of the natural numbers $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ which are no greater than 1000, and arrange them in increasing order.

14. Find all pairs of powers in the previous list who have differences no greater than 10.