PERCENTAGES

WARM UP!

The Math Circle instructors are baking cakes for a bake sale, but their recipes call for measurements which they can't seem to measure with their tools. Help them find a way to correctly fulfill the recipes!

- Morgan and Pax are in charge of timing the oven. They need a way to measure 15 minutes, but they only have a 7-minute hourglass and an 11-minute hourglass.

  - Start timing both of them.
  - After 7 minutes, the 7-minute glass runs out. Flip over the 7 minute glass.
  - After 4 more minutes, the 11-minute hour glass runs out. Flip over the 7 minute glass again.
  - After 4 more minutes, the 7-minute hour glass will run out. It has now been 7 + 4 + 4 = 15 minutes.

- Derek and Dustin need exactly 6 gallons of water, but they only have a 9-gallon bucket and a 4-gallon bucket. (the buckets do **not** have measurement markings on the sides, and there is an infinite source of water)

  - Fill the 9-gallon bucket with water.
  - Dump it into the 4-gallon bucket. We now have 5 gallons in the 9-gallon bucket.
  - Dump out the 4-gallon bucket (into the sink or something). Pour 4 gallons from the 9-gallon bucket into the 4-gallon bucket.
  - Dump out the 4-gallon bucket again. Pour the remaining 1 gallon into the 4- gallon bucket.
  - Fill up the 9-gallon bucket, pour three gallons into the 4-gallon bucket to fill it up. Now there are 6 gallons in our 9-gallon bucket.

<div align="center">PROBLEMS</div>

(1) Fill in the blanks in the following two sentences:

    (a) To increase a number by 31% means to multiply it by **(1 + .31)**.

    (b) To decrease a number by 31% means to multiply it by **(1 - .31)**.

(2) A store secretly increased the price of their sweaters by 100%. The next day, they had a special 50% off sale on sweaters. How does the sale price compare to the original price (before they increased it by 100%)?

- First the price raised by 100%, so it is now twice as much. Taking 50% off makes the price half as much as it was, this means the price is exactly the same as the original.

(3) Pax and Derek have a bit of a junk food addiction.

    (a) Last week, Pax ate 20% more junk food than what he ate two weeks ago. This week, Pax will eat 20% less junk food than what he ate last week. Does Pax eat more junk food this week or two weeks ago?

    (b) Last week, Derek at 20% less junk food than what he ate two weeks ago. This week, Derek will eat 20% more junk food than what he ate last week. Does Derek eat more junk food this week or two weeks ago?

- This is exactly the same in both cases: *An increase by 20% is to multiply by (1 + .2) and a decrease is to multiply by (1-.2) since multiplication is commutative it doesn't matter what order we do this. The result is the original times* $(1 + .2)(1 - .2) = .96$ *so they eat more junk food two weeks ago.*

(4) The price of gas dropped by 20%. How much more gas can you buy for the same amount of money? (answer in percentage)

- Lets have G denote the original price, and M denote how much money we have. So originaly we can buy $\frac{M}{G}$ gas. A decrease by 20% means the current price is $\frac{4}{5}G$. This means we can now buy

$$\frac{M}{\frac{4}{5}G} = \frac{5}{4}\frac{M}{G}$$

gas. This corresponds to an increase by 25%.

(5) Dustin and Derek were competing to sell candy. Derek offered the following deal: candy is 15% off, and if you are from the Math Circle you get an extra 15% off your total. Dustin offered 30% off. Originally Dustin and Derek sold candy for the same price, who is giving the better deal?

- Let P be the originally price of a piece of candy. Derek is now charging $P(.85)^2$ and Dustin is now charging $P(.7)$, it turns out $.85^2 > .7$, so Dustin is giving the better deal.

(6) Morgan's favorite item was on sale for \$105 at the store. It usually costs \$150. What is the percent discount?

- $\frac{105}{150} \cdot 150 = 105$, just looking at this equation we can see the discount is $\frac{45}{150}$, which is 30%.

(7) At the beginning of the year, 40% of the Math Circle students were boys. After 6 more boys joined the Math Circle and 5 girls left the Math Circle, the number of girls and boys was equal. How many people were at the Math Circle in the beginning of the year.

- Let B be the initial number of boys and G the initial number of girls. Based on the information we have two equations:

$$B + 6 = G - 5$$
$$.4(B + G) = B$$

Solving this linear system yields 22 boys and 33 girls for a total of 55 students.

(8) This year, Shack (a basketball player with no affiliation to Shaq) shot 30% more freethrows than last year. However, this year he made only 60% of his freethrows, while last year he made 80%. Do you think he made more freethrows this year or last year?

- Lets set S as the number of attempts last year and M as the number of makes. Then we will have $\tilde{S}$ and $\tilde{M}$ represent the same values for this year. By the first assumption we have

$$\tilde{S} = S(1 + .3)$$

And by our assumption that he made 60% of his freethrows we have

$$\tilde{M} = S(1 + .3)(.6) = S(.78)$$

and similarly

$$M = S(.8)$$

Therefore we see that $M > \tilde{M}$.

(9) Dustin is an avid cyclist, however, during some seasons he has more time to ride than others. Because of this his weight fluctuated throughout the year. In spring his weight dropped by 25%. In summer, it increased by 20%. In fall, it dropped 10%. In winter, it increased 20%. Do you think Dustin lost or gained weight?

- Let W be his initial weight. After the year his weight becomes:

$$W(1 - .25)(1 + .2)(1 - .1)(1 + .2) = W(.972) < W$$

so we see he lost weight.

(10) Deven made some poor investments. He initially invested 7,500 dollars. The total value of his investments dropped twice by the same percentage and he ended up with 4,800 dollars. What was the percentage?

- Let x be the percentage divided by 100. Then we have $7500(1-x)^2 = 4800$ or $(1-x)^2 = \frac{4^2}{5^2}$ so $1-x = \frac{4}{5}$ and $x = \frac{1}{5}$. Thus the percentage is 20%.

(11) (**review**) Write $.370370370\cdots$ as a fraction in lowest terms.

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$$.370370370\cdots = \frac{370}{999} = \frac{37 \cdot 10}{37 \cdot 27} = \frac{10}{27}.$$

(12) (**review**) Write $\frac{51.2}{21.6}$ as a fraction (without any decimals) in lowest terms.

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$$\frac{51.2}{21.6} = \frac{512}{216} = \frac{256}{108} = \frac{128}{54} = \frac{64}{27}$$

(13) You start with the number 51.2 and increase it three times by the same percentage. Then you decrease it three times by the same percentage. You end up with the number 21.6. What is the percentage?

- Again let x be the percentage divided by 100. Then, after all our increasing and decreasing we have $51.2(1-x)^3(1+x)^3 = 21.6$ by the previous problem we have $(1-x)^3(1+x)^3 = \frac{27}{64}$ inspection shows that $x = \frac{1}{2}$ works. So the percentage is 50%.

(14) Derek and Pax live in the same building and leave for the Math Circle at the same time. Pax is a little taller, so his steps are 10% longer than Derek's. However, Pax also takes 10% fewer steps per second than Derek. Who arrives at Math Circle first?

- Let P be Pax's step length and $\tilde{P}$ be the number of steps Pax takes per second. We know that Pax's speed is then $P\tilde{P}$. Let D and $\tilde{D}$ represent the same values for Derek. By our assumption we have

$$P = \frac{11}{10}D$$
$$\tilde{P} = \frac{9}{10}\tilde{D}$$

Therefore, we have

$$P\tilde{P} = \frac{99}{100}D\tilde{D} < D\tilde{D}$$

so we know that Derek is moving faster, and hence arrives at Math Circle first.

HOMEWORK!

(1) Which is bigger: 27.13% of 13.27 or 13.27% of 27.13?

(2) Morgan is a much wiser investor than Deven. He initially invested 1000 dollars. His total increased twice by the same percentage. In the end, he had 1210 dollars. What was the percentage?

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