PERMUTATIONS - II

JUNIOR CIRCLE 11/17/2013

Operations on Permutations. Among all the permutations of n objects one stands out as the simplest: all the objects stay in their places. This permutation is called the *identity permutation* (or the *trivial permutation)*.

Given two permutations, we can find their composition. Composition of permutations is a new permutation which is obtained by performing the first permutation followed by the second permutation.

(1) Find composition of the following two permutations by using both your cards and braids:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}$$

(2) Find composition of the following composition of permutations by using both your cards and braids:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}$$

(3) What do you get if you compose any permutation with an identity permutation?

(4) What do you get if you compose an identity permutation with an arbitrary permutation?

Another operation we can do with a permutation is finding its inverse. Given a permutation, the *inverse permutation* has the effect of returning the objects to the original order. This means that composition of a permutation and its inverse is the identity permutation.

(5) Check (using braids or cards) that the inverse of

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

(6) Find the inverse of the following permutations (use your cards and/or braids):

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 4 \end{pmatrix}, \qquad \text{inverse=}\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 4 & 3 \end{pmatrix}, \qquad \text{inverse=}\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 2 & 1 \end{pmatrix}, \qquad \text{inverse=}\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}$$

Switching Positions vs. Switching Cards.

(1) Play the following game with your partner several times:
- Take 5 cards with numbers $1, 2, 3, 4, 5$ written on them;
- Mix the order of the cards and put them on the table in the new order;
- Ask your partner to return the cards to the original order $(1, 2, 3, 4, 5)$ by repeating the following operation:

(a) *switch card in position* 1 *with a card in another position;*

Do you think you can always return to the original order this way?

(b) *switch card with number* 1 *written on it with any other card*;

Do you think you can always return to the original order this way?

Compare the games in parts (a) and (b).

(2) Let's look at how the order of the cards on the table corresponds to a permutation.

(a) Suppose that we have the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

Write down the order of the cards on the table that corresponds to this permutation:

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Draw the braid that corresponds to this permutation below:

(b) Suppose the cards are put on the table in the following order:
[2] [3] [1] [4]

Write down the permutation that produces this order of the cards starting from the standard order (1, 2, 3, 4):

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}$$

Draw the braid that corresponds to this permutation below:

(3) Suppose the cards are in the order $(3, 2, 4, 1)$. How can we get back to the original position $(1, 2, 3, 4)$ by performing several switches with position 1 (i.e., with the card that is at the left end)?

Model this with the following:

- First, switch cards in positions [1] and [ ];
- Second, switch cards in positions [1] and [ ];
- Third, switch cards in positions [1] and [ ];
- Forth, switch cards in positions [1] and [ ];

Note: You may not need to use all 4 of these steps, you can use less and still put the numbers into the original order.

(4) Suppose the cards are in the order $(3, 2, 1, 5, 4)$. How can we get back to the original position $(1, 2, 3, 4, 5)$ by performing several switches with position 1?

- Model this with the following:
  - First, switch cards in positions $\boxed{1}$ and $\boxed{}$;
  - Second, switch cards in positions $\boxed{1}$ and $\boxed{}$;
  - Third, switch cards in positions $\boxed{1}$ and $\boxed{}$;
  - Forth, switch cards in positions $\boxed{1}$ and $\boxed{}$;
  - Fifth, switch cards in positions $\boxed{1}$ and $\boxed{}$;
  - Sixth, switch cards in positions $\boxed{1}$ and $\boxed{}$;

  Note: You may not need to use all 6 of these steps, you can you less and still put it into the original position.

Composition of permutations is NOT Commutative.

(1) Find composition of the following two transpositions:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}$$

- Model this with cards to figure out where the cards move to after the two switches:

$$\begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ \boxed{} & \boxed{} & \boxed{} \\ \downarrow & \downarrow & \downarrow \\ \boxed{} & \boxed{} & \boxed{} \end{array}$$

(2) Find composition of the following two transpositions:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}$$

- Compare the results from 1 and 2. Are they the same? Why or why not? Does the order in which you compose these two permutations matter?

(3) Find the results of the following two compositions of permutations:

(a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}$$

(c) Compare the results. Are they the same? Why or why not?

(d) Does the order in which you perform these two permutations matter?

(e) What is the difference between this problem and the previous one?