THE PROOF IS IN THE PICTURE

(an introduction to proofs without words)

LAMC INTERMEDIATE GROUP - 11/24/13

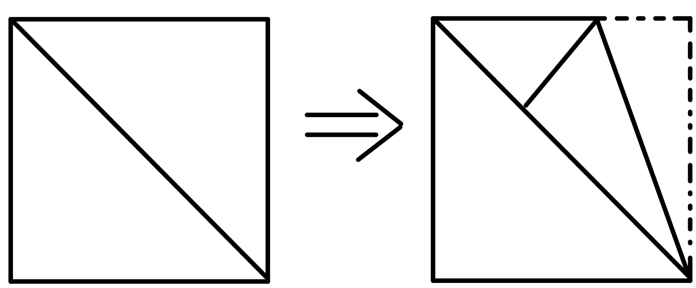
WARM UP (PAPER FOLDING)

Theorem: The square root of 2 is irrational.

We prove this via the mathematical technique of contradiction. To that end, suppose that the square root of 2 was rational. This means that we can find a right isosceles triangle with integer sides.

Follow these steps to prove this is impossible:

- Take out a square piece of paper. Suppose your square has side-length n units and that the diagonal has length m units (n and m integers).
- Fold it in half along the diagonal so that you have a right isosceles triangle.
- Fold a leg of your triangle onto the diagonal. (See the figure below)
- Once you have finished folding your triangle, flip to the next page to work out the details.

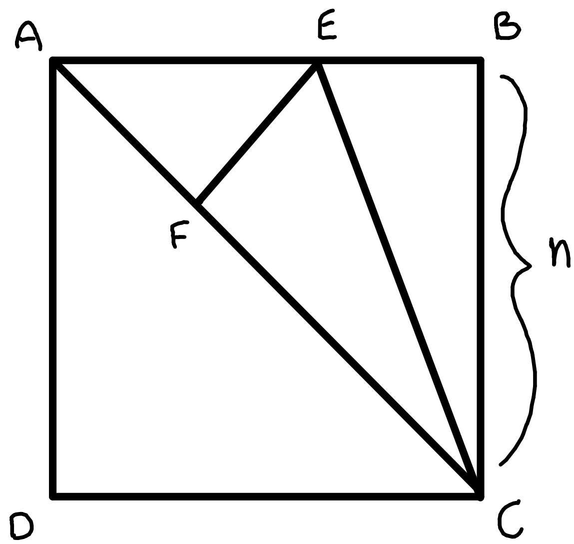


This proof originally by John Conway

Recall the following facts:

- The sum of the degrees of the angles in a triangle is 180.
- In a right triangle the side lengths are related to the angles. In particular if the two non-right angles are the same you have an isosceles triangle.

Use these facts to fill in the missing lengths: CF, FA, EF, EB and EA in the following picture.

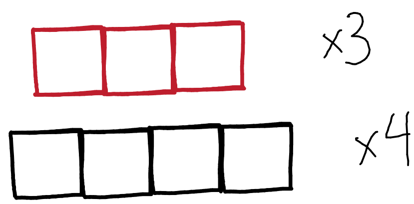


You have now shown that triangle AFE is a right isosceles triangle. Find the contradiction.

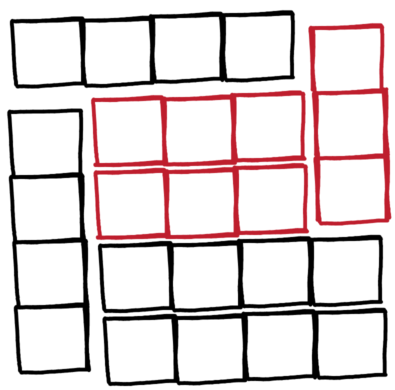
PROOFS BY PUZZLE

For each problem in this section, you are given a theorem and puzzle pieces. The goal is to arrange the puzzle pieces in some fashion to prove the theorem. Here is an example (my picture has no explanation, but please explain why your pictures are proofs when you solve the subsequent problems!)

- Question: Prove that $3^2 + 4^2 = 5^2$ using the following puzzle pieces:



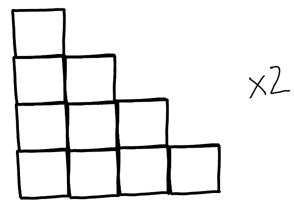
- Solution:



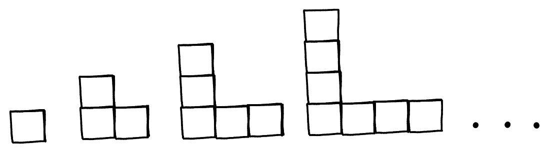
(1) The triangular number t_n is defined as

$$t_n = 1 + 2 + \cdots + n.$$

Prove that $t_n = n(n+1)/2$ given the following puzzle pieces:



(2) Prove that the sum of the first n odd numbers $1 + 3 + \cdots + (2n - 1) = n^2$ using the following puzzle pieces:

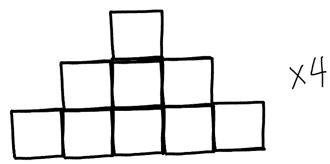


Draw separate pictures for $n = 2, n = 3$, and $n = 4$ then explain how exactly to generalize for any number n.

(3) In this problem, we prove the same theorem as in problem 2 using different puzzle pieces. Prove that the sum of the first n odd numbers

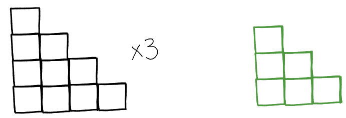
$$1 + 3 + \cdots + (2n - 1) = n^2$$

using the following puzzle pieces:

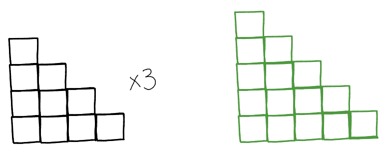


(4) In this problem you will prove two related identities.

- Prove that $3t_n + t_{n-1} = t_{2n}$ using the following puzzle pieces:



- Prove that $3t_n + t_{n+1} = t_{2n+1}$ using the following puzzle pieces:

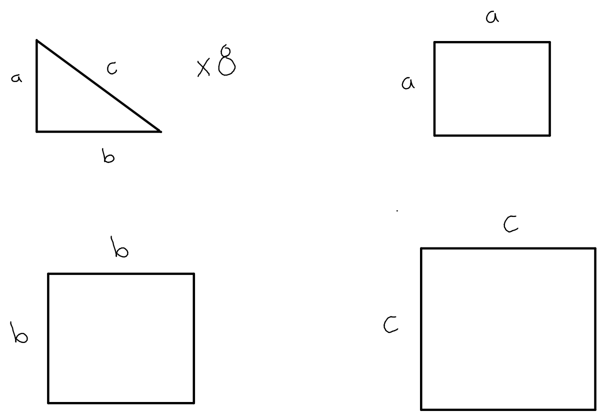


(5) (Challenge) The goal is to prove the identity $t_{(n-1)}^2 + t_n^2 = t_{n^2}$.

- Prove that $t_1^2 + t_2^2 = t_{2^2}$ using the following puzzle pieces:

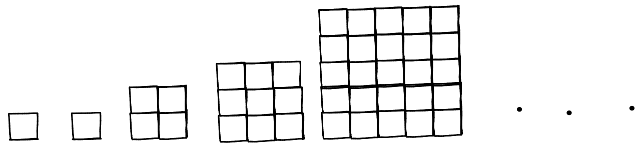
- Find your own puzzle pieces to prove that $t_2^2 + t_3^2 = t_{3^2} = t_9$ and explain how to generalize this to any number n.

(6) Prove the Pythagorean theorem using the following puzzle pieces:



Hint: You should draw a big square (with side-length $a + b$) in two different ways

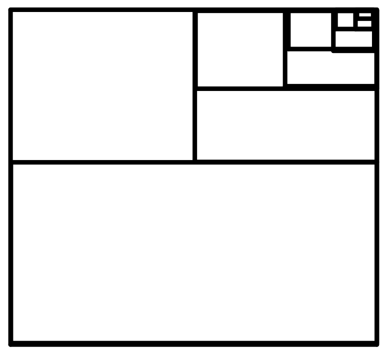
(7) Prove that $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$ where F_n is the n^{th} Fibonacci number using the following puzzle pieces:



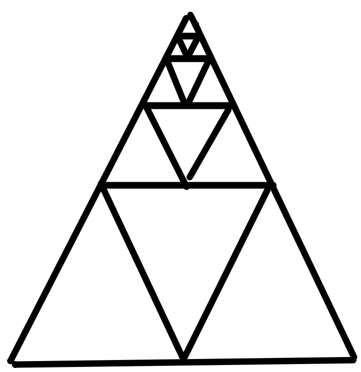
Draw separate pictures for $n = 1, n = 2, n = 3$ and $n = 4$ then explain how exactly to generalize for any number n.

EVALUATE THE INFINITE SUMS!

(1) Evaluate the infinite sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ using the following picture:



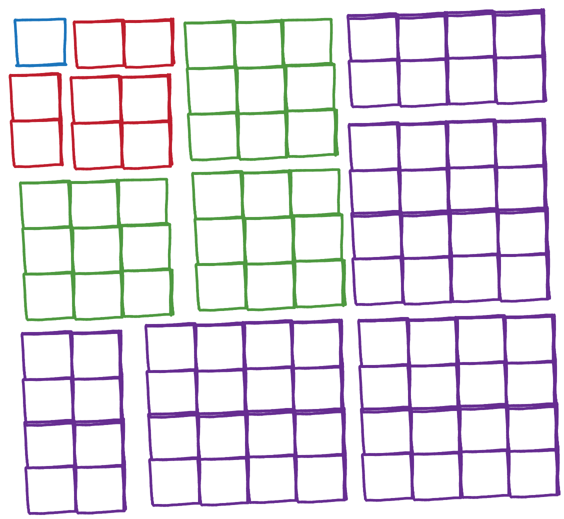
(2) Evaluate the infinite sum $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$ using the following picture:



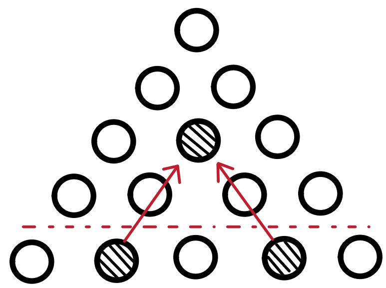
Picture Proof Jeopardy

In this section, we present a picture proof but don't tell you the theorem it is proving. See if you can figure out what is going on in the picture and provide the theorem it proves.

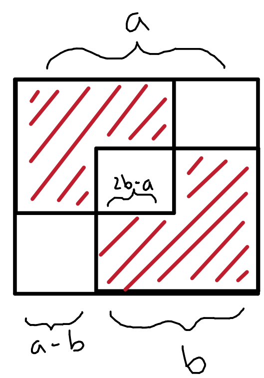
(1) Hints: Figure out the length and width of this rectangle. Now you can write the area as either length times width or as the sum of the areas of the inside pieces.



(2) Hint: This is about a correspondence between two things you can count.



(3) (Challenge) This is another proof of the irrationality of the square root of two (due to Tennenbaum). Try to explain the picture.



Hint: Start with $\sqrt{2} = \frac{a}{b}$ with a and b as small as possible. Note that $\sqrt{2} > 1$ so we have $a > b$. Now the picture makes sense, try to use it to find a smaller fraction representation for $\sqrt{2}$.