LAMC Beginners' Circle **November 24, 2013**

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Warm-up

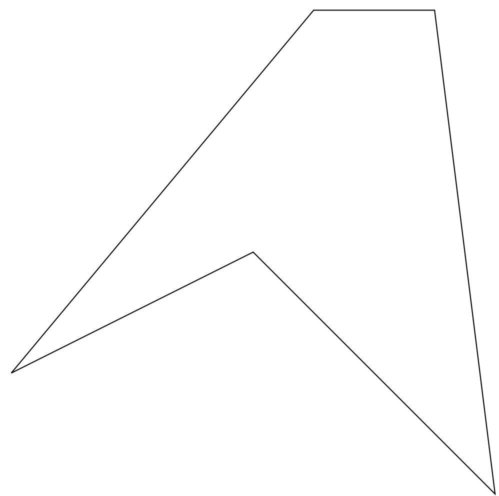
Problem 1 *Transform the following hexadecimal number to the octal form without switching to the decimals.*

$$b29_{16} =$$

Problem 2 *What is the hexadecimal value of the following decimal number?*

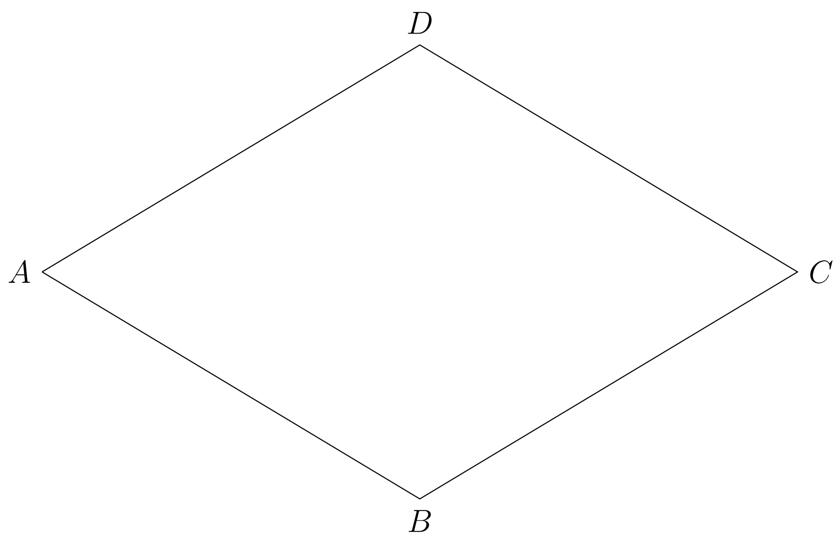
$$3,595_{10} =$$

Problem 3 *Use a compass and a ruler to draw a polygon congruent to the given one in the space below.*



Back to the mini-course

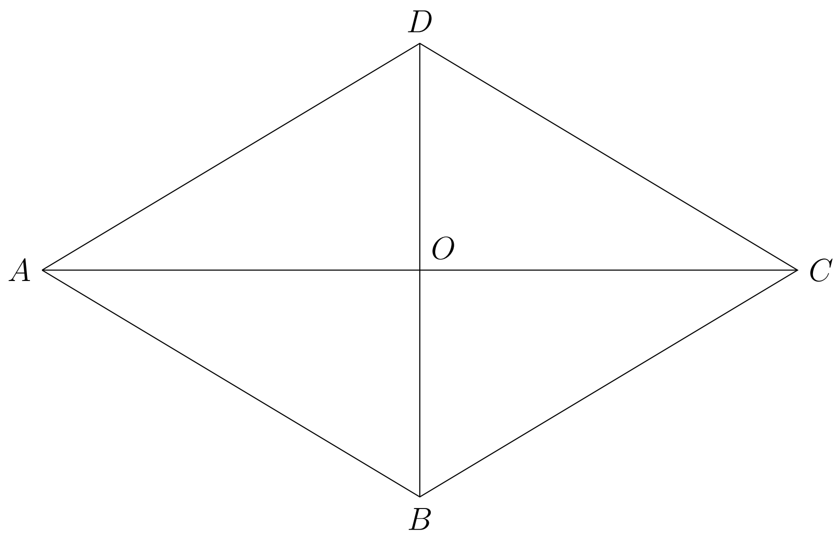
Problem 4 *To prove that a rhombus is a parallelogram, fill out the reason part of the chart below.*



Claim	**Reason**
$\vert AB\vert = \vert CD\vert$ *and* $\vert AD\vert = \vert BC\vert$	
$ABCD$ *is a parallelogram.*	

Q.E.D.

Problem 5 *To prove that diagonals of a rhombus intersect at the right angle, consider the following picture.*



Then fill out the reason part of the following chart.

Claim	**Reason**
$\lvert AB\rvert = \lvert BC\rvert = \lvert CD\rvert = \lvert DA\rvert$	
$\lvert AO\rvert = \lvert OC\rvert$ *and* $\lvert BO\rvert = \lvert OD\rvert$	

Continues on the next page.

Claim	**Reason**
$\triangle ABO \cong \triangle BCO \cong$ $\triangle CDO \cong \triangle DAO$	
$\angle AOB \cong \angle BOC \cong$ $\angle COD \cong \triangle DOA$	
$\measuredangle AOB = \measuredangle BOC =$ $\measuredangle COD = \measuredangle DOA = 90^\circ$	

Q.E.D.

Definition 1 *Two straight lines intersecting at the right angle are called orthogonal, or perpendicular, to each other.*

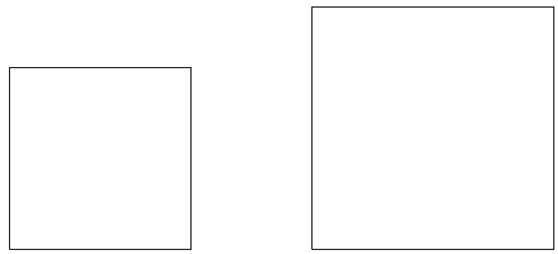
If the lines l and s are perpendicular, we will denote it the following way.

$$l \perp s$$

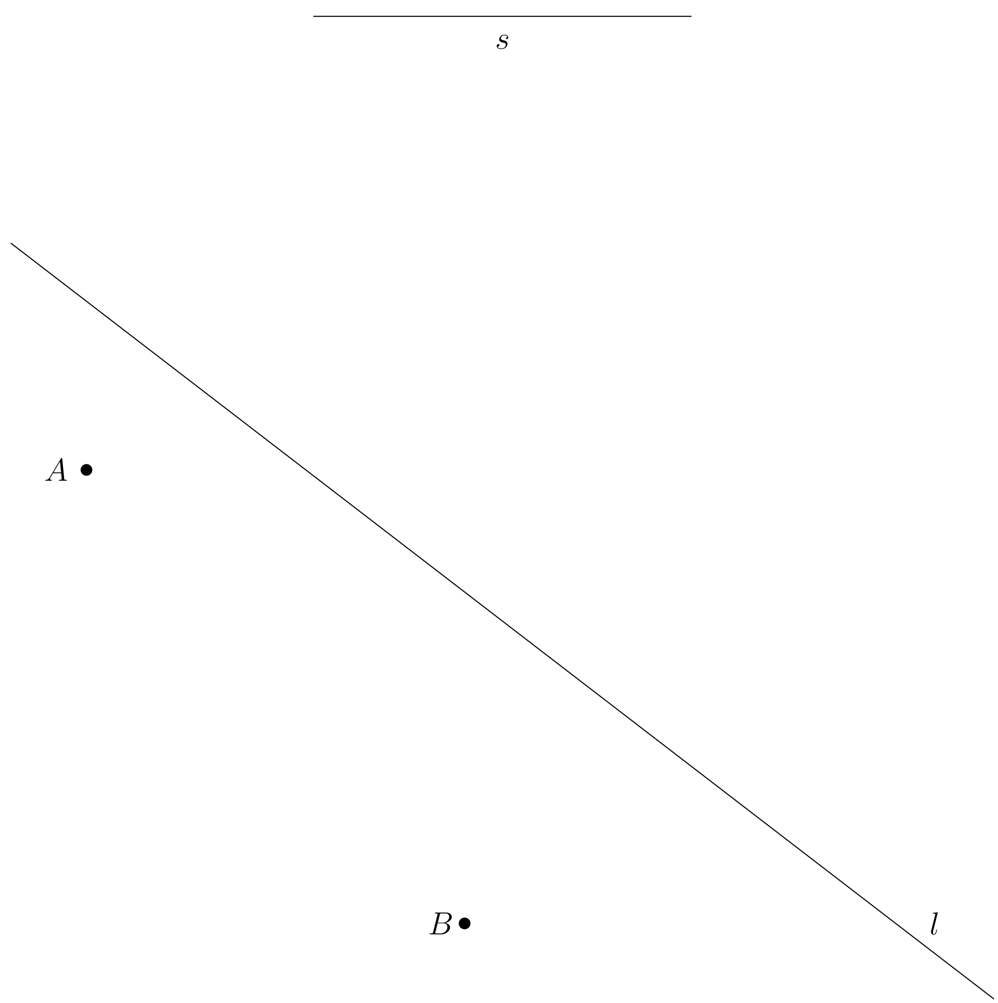
According to Problem 5, the diagonals of a rhombus are always orthogonal. This gives us a way to construct orthogonal lines with a compass and a ruler.

Problem 6 *Use a compass and a ruler to draw a square having the following diagonal in the space below.*

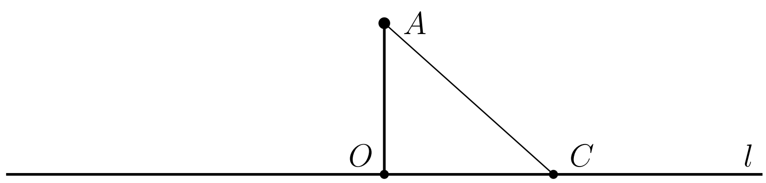
Problem 7 *Given two squares on the picture below, use a compass and a ruler to construct the third square such that its area equals to the sum of the areas of the given squares.*



Problem 8 *Given a segment s, two points, A and B, and a straight line l below, use a compass and a ruler to construct the shortest path connecting A to B and going for the distance $|s|$ along l.*



Problem 9 *Consider the picture below. Let $(AO) \perp l$. Let C be any point on l different from O.*



To prove that $|AC| > |AO|$, fill out the reason part of the chart below.

Claim	**Reason**
$\|AC\|^2 = \|AO\|^2 + \|OC\|^2$	
$\|AC\| > \|AO\|$	

Q.E.D.

We just have proven the following.

Theorem 1 *In the Euclidean plane, the shortest path from a point to a straight line is a perpendicular from the point to the line.*

We still need to prove that the perpendicular is unique; a perpendicular is in fact the perpendicular.

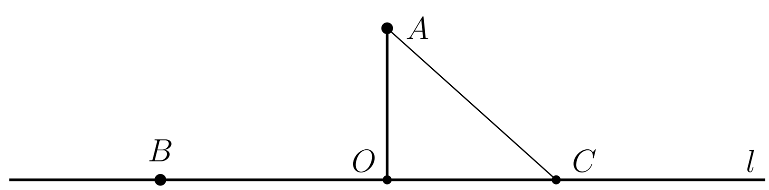
Postulate 1 (Euclid) *A straight line segment can be drawn joining any two points.*

Note that Euclid does not specify what object the points belong to. He neither requires the line to be unique. It is traditional to modify Postulate 1 as follows.

Postulate 1, modified. *For any two different points in a plane, or any higher-dimensional Euclidean space, there exists a unique straight line passing through them.*

Theorem 2 *For any point and straight line in the Euclidean plane, there exists a unique straight line passing through the point orthogonal to the original line.*

Proof — Assume that both lines AO and AC are orthogonal to the line l on the picture below.

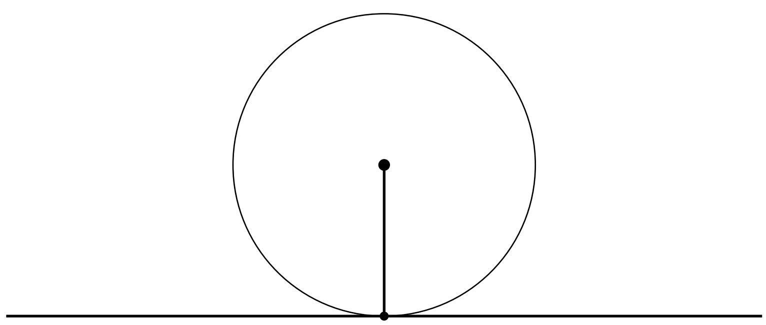


Problem 10 *To finish the proof, fill out the reason part of the chart below.*

Claim	**Reason**
$\measuredangle BOA = \measuredangle OCA = 90^\circ$	
$(AO) \parallel (AC)$	

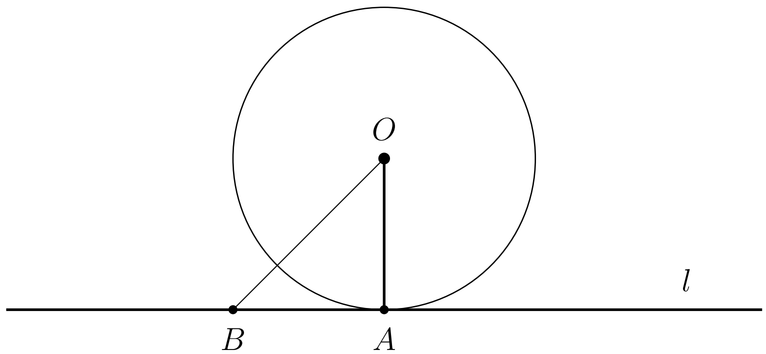
Parallel lines AO and AC cannot intersect at point A. We arrive to a contradiction. Thus, the initial assumption must have been wrong. Q.E.D.

Definition 2 *A straight line is called tangent to a circumference, if they intersect at one point.*



Theorem 3 *A line tangent to a circumference is orthogonal to the radius drawn from their common point to the circumference center.*

Proof — Assume the opposite. According to Theorem 2, there exists a unique perpendicular from O to l. It is not OA. Let it be OB.

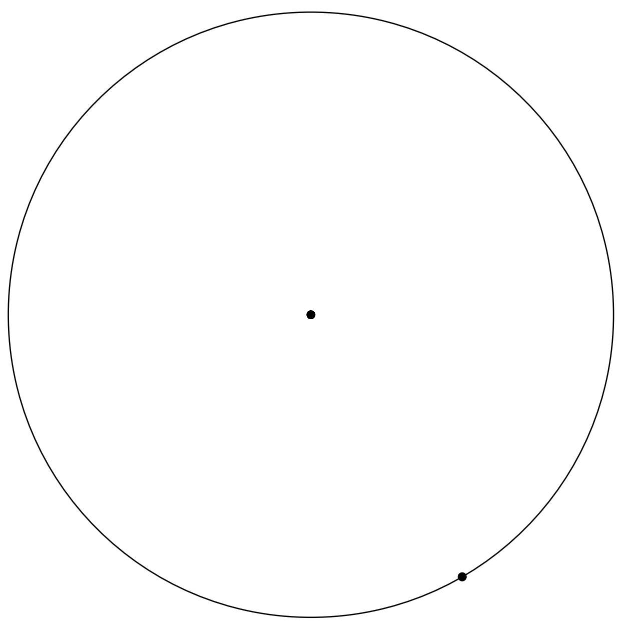


Problem 11 *To finish the proof, fill out the reason part of the chart below.*

Claim	**Reason**
Point B lies outside of the circumference.	
$\lvert OB\rvert > \lvert OA\rvert$	
$\lvert OB\rvert < \lvert OA\rvert$	

The resulting contradiction finishes the proof. $\square$

Problem 12 *Use a compass and a ruler to construct a straight line tangent to the given circumference at the given point.*



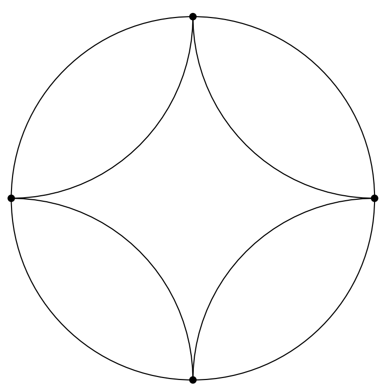
The area A of a circle of radius r in the Euclidean plain is given by the following formula.

$$A = \pi r^2, \quad \pi = 3.141592653... \tag{1}$$

Problem 13 *Estimate the area of a circle of radius $r = 2'$.*

$$A =$$

Problem 14 *The four points on the picture below split the circle into four congruent arcs. The four inner arcs joining the points are all congruent to one another as well as to the outer arcs. What is the area of the star figure in the center of the circle?* [1]



$$A =$$

[1] Thanks to Samir's mom, Sarita, for e-mailing this problem to us.

Problem 15 *Prove that in the Euclidean plane, a straight line cannot intersect all the three sides of a triangle.*

Problem 16 *One chooses* $n+1$ *numbers between* 1 *and* $2n$*. Show that she/he has selected two numbers* a *and* b *such that* a *divides* b*.*