CONTINUITY PROBLEMS IN GEOMETRY II (UCLA MATH CIRCLE)

The idea of using continuity is one of the oldest and ubiquitous in mathematics in general, and in geometry in particular. Below we assembled several problems where it plays a crucial role.

Note: Stars indicate harder problems.

Introductory Problem 1. Let C be the boundary of a square in the plane $\mathbb{R}^2$. Let z be a point on C. Prove that C has an *inscribed equilateral triangle* with one vertex at z. In other words, prove that there exists points x, y on C such that $\triangle xyz$ is equilateral.

Problem 2. Same as Problem 2, but $\triangle xyz$ has angles $\angle xyz = \angle yxz = 30°$, and z is not a corner.

Problem 3. Same as Problem 2, but $\triangle xyz$ has angles $\angle xyz = \angle xzy = 30°$.

Problem 4. Let Q be a triangle in the plane, z be a point inside. Prove that z is a midpoint of two points on Q.

Problem 5. Let Q be a triangle in the plane, z be a point inside. Prove or disprove: there are two points x, y on Q such that z divides interval $[xy]$ with ratio 10:1.

Problem 6*. Same as Problem 5, but the ratio is 2:1.

Problem 7. Let L_1, L_2, L_3 be three distinct parallel lines in the plane $\mathbb{R}^2$. Prove that there exists an equilateral triangle $\triangle x_1x_2x_3$ with a vertex x_i on the line L_i, $i = 1, 2, 3$.

Problem 8. Let L_1, L_2, L_3 be three distinct parallel lines in the plane $\mathbb{R}^2$. Prove that there exists a triangle $\triangle x_1x_2x_3$ with a vertex x_i on the line L_i, $i = 1, 2, 3$, and $\angle x_1x_2x_3 = \angle x_2x_1x_3 = 45°$.

Problem 9*. Let $H_1, \ldots, H_4$ be four distinct parallel planes in the space $\mathbb{R}^3$. Prove that there exists a regular tetrahedron with one vertex on each plane H_i.

Problem 10. Let Q be a convex polygon in the plane $\mathbb{R}^2$. Prove that there is a circumscribed square S, i.e. square S which touches Q on all four sides.

Problem 11. Let Q be a convex polygon and ℓ is a line in the plane $\mathbb{R}^2$. Prove that Q has an inscribed rhombus with a diagonal parallel to ℓ.

Problem 12*. Let Q be a convex polygon in the plane $\mathbb{R}^2$. Prove that C has an *inscribed square.*

Problem 13*. Let C be a closed polygonal curve in the space $\mathbb{R}^3$ and z a point on C. Prove that C has an *inscribed equilateral triangle* with one vertex at z.

P.S. For more on inscribed polygon problems, see Chapter 5 in
Igor Pak, *Lectures on Discrete and Polyhedral Geometry*,
available online at `http://www.math.ucla.edu/~pak/book.htm`

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