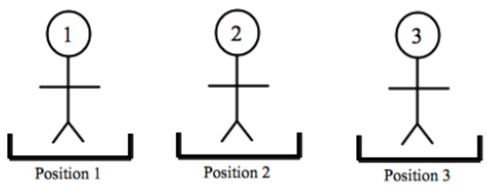
HANDOUT 7: PERMUTATIONS

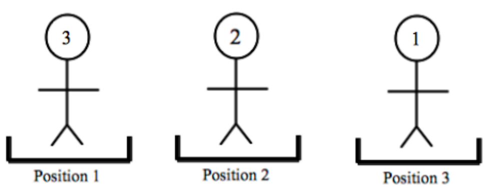
JUNIOR CIRCLE 11/7/2013

Several people (with labels $1, 2, 3\ldots$ attached to the back of their T-shirts) are standing in positions $1, 2, 3\ldots$ at the beginning:



Suppose that they switch positions. For example, let's say that

- people in positions 1 and 3 switch positions
- person in position 2 remains there



We can write this as follows:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{pmatrix}.$$

The first row represents the starting positions. The second row represents the end positions. To make the notation shorter, we will not be drawing arrows, so that the above looks like

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

A *permutation* of a set of objects is an arrangement of those objects in a particular order. In this case, this is called a *permutation* because we are permuting (or changing) the order in which people stand.

(1) Suppose that we have the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

(a) What positions did not shift?

(b) Where did position 3 move to?

(c) What position moved into position 3?

(2) In a row of 5 people, the middle person stayed in his place. The two people at the ends of the row switched places. Their neighbors also switched places. Indicate where each of the positions moved to:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{pmatrix}$$

(3) In a row of 5 people, all of the odd positions didn't change. The two even positions switched. Indicate where each of the positions moved to:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{pmatrix}$$

(4) In a row of 6 people, the first 5 people shifted one place to the right, and the last person moved to the first place. Write this down as a permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & & \end{pmatrix}$$

(5) Compare the following three permutations. Which of them would you call more "mixed-up"? Why? (Hint: Circle the positions that changed)
- The first permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix};$$

- The second permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix};$$

- The third permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix};$$

(6) The simplest possible permutation just switches two positions (and keeps the rest the same). Such permutations are called *transpositions*.

Give several examples of transpositions below:
(a)

$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

Give an example of a permutation that is NOT a transposition:

(a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

It turns out that transpositions are the building blocks of permutations. By performing several transpositions in a row, you can get various permutations. The amazing fact is that you can get any permutation this way.

(7) In a row of three people,
   (a) Write down the permutation that switches positions 1 and 2;

   (b) Then, write down the permutation that switches positions 2 and 3;

   (c) Perform these two permutations (first a, then b) using sticky notes and record the end result:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}.$$

(8) In a row of three people,
   (a) Write down the permutation that switches positions 2 and 3;

   (b) Then, write down the permutation that switches positions 4 and 2;

   (c) Finally, write down the permutation that switches positions 1 and 3:

   (d) Perform these three permutations (first a, then b and finally c) using sticky notes and record the end result:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}$$

(9) Play the following game several times:
- select several cards (e.g., 4);
- place them on the table in the usual order $(1,2,3,4)$;
- exchange two cards (e.g., switch 2 and 4);
- exchanged two other cards;
- repeat the previous step several times (as many as you want to);
- record the permutation you get below:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix};$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix};$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}.$$

(10) Play the following game with your partner:
- Take 4 cards with numbers $1,2,3,4$ written on them;
- mix up the order of the cards in some way (any way you like);
- ask your partner to get this permutation from to original order by perfoming several switches.
- Do you think you can always do this?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix};$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}.$$

(11) How can we get the following permutation by performing several switches?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 2 & 1 & 3 \end{pmatrix};$$

- Model this with the following:
  - First, switch positions $\square$ and $\square$;
  - Second, switch positions $\square$ and $\square$.
- Can you find another way?
  - First, switch positions $\square$ and $\square$;
  - Second, switch positions $\square$ and $\square$.

(12) In problem 11, we performed one permutation followed by another permutation. Let's do something like this again.

- Suppose you first perform the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{pmatrix}.$$

- Then, you perform the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix}.$$

- Model this with cards to figure out where the positions move after the two switches:

$$\begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ \square & \square & \square \\ \downarrow & \downarrow & \downarrow \\ \square & \square & \square \end{array}$$

- If we forget about the intermediate step, this can be written as:

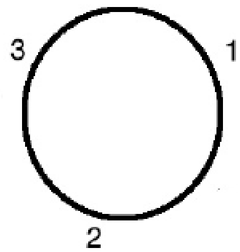
$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}.$$

- To show the result of these two permutations performed one after the other we write:

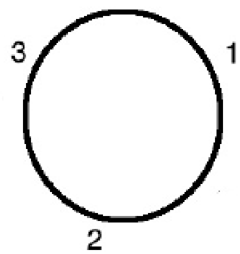
$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}$$

(13) Find the result of performing the following two permutations one after the other:

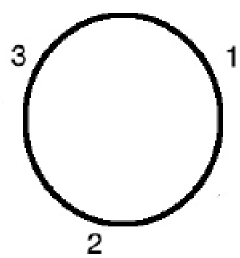
$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}.$$



- If 3 people are sitting around the tables, what does the first permutation represent? (Use arrows to show where the person moves.)



- What does the second permutation represent? (Use arrows to show where the person moves.)

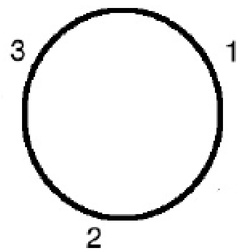


- What is the result of doing these 2 operations one after the other?

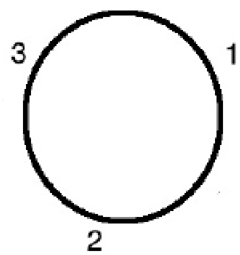
- Does this agree with your answer above?

(14) Find the result of performing the two permutations in a row:

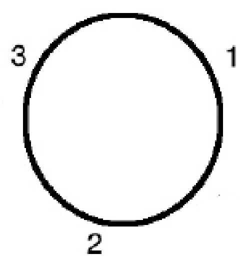
$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}.$$



- If 3 people are sitting around the tables, what does the first permutation represent? (show with arrows)



- What does the second permutation represent? (show with arrows)



- What is the result of doing these 2 operations one after the other?

- Does this agree with your answer above?

HOMEWORK: Combine 3 transpositions to get a permutation. Draw a picture with braids to help you check your work.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{pmatrix}$$

Place your answer on an index card so that you can switch with a partner next week as a warm-up.