COMPLEX NUMBERS: A PRIMER

MATH CIRCLE (HS1) 11/03/2013

When we first learn about square roots, we learned that we cannot take the square root of a negative number.

But what if we could? Suppose we define $\sqrt{-1} = i$. We call a number *imaginary* if it is a multiple of i.

We call the sum of a real number and an imaginary number a *complex number*. Therefore, complex numbers are of the form $a + bi$ for $a, b \in \mathbb{R}$ (i.e. a, b are real numbers). If $z = a + bi$ is a complex number, we call a the real part of z (denoted $a = \text{Re}(z)$), and we call b the imaginary part of z (denoted $b = \text{Im}(z)$).

Basic Arithmetic For Complex Numbers

We can add and subtract complex numbers by pretending i is a variable. For example,

$$(1 + i) + (-3 + 4i) = -2 + 5i \text{ and } (3 - 3i) - (-2 - 5i) = 5 + 2i.$$

We can multiply complex number by first pretending that i is a variable and then using the fact that $i^2 = -1$. For example,

$$(1+i)\cdot(-3+4i) = -3+i+4i^2 = -7+i \text{ and } (3-3i)\cdot(-2-5i) = -6-9i+15i^2 = -21-9i.$$

1) Simplify each expression:

a) $(1 + i) + (1 + 3i)$

b) $(1 + 2i) - 5i$

c) $i(3 + i)$

d) $(1 - i)(1 + i)$

e) $(2 + 3i)(3 + 2i)$

2) Suppose $x = a + bi$ and $y = c + di$.

a) Write a formula for $x + y$. b) Write a formula for $x \cdot y$.

Complex Conjugation

Let $z = a + bi$ be a complex number. The *complex conjugate* of z (denoted $\overline{z}$) is defined by

$$\overline{z} = a - bi.$$

The *modulus* (or absolute value) of z (denoted $|z|$) is defined by

$$|z| = \sqrt{a^2 + b^2}.$$

Some problems adapted from previous Math Circle handouts.

For example, suppose $x = 5i$ and $y = 2 + 3i$, then

$$\overline{x} = -5i,\ |x| = \sqrt{0^2 + 5^2} = 5 \text{ and } \overline{y} = 2 - 3i,\ |y| = \sqrt{2^2 + 3^2} = \sqrt{13}.$$

3) Prove that $z \cdot \overline{z} = |z|^2$.

4) Suppose $x = a + bi$ and $y = c + di$.

a) Prove that $\overline{x + y} = \overline{x} + \overline{y}$. b) Prove that $\overline{x \cdot y} = \overline{x} \cdot \overline{y}$.

The complex conjugate is useful in dividing complex numbers. For example,

$$\frac{1+i}{2+i} = \frac{(1+i)(2-i)}{(2+i)(2-i)} = \frac{3+i}{5} = \frac{3}{5} + \frac{1}{5}i.$$

5) Simplify the following:

a) $\overline{\overline{(1+i)}(1+i)}$

b) $|1 + i|$

c) $|3 - 4i|$

d) $25/(3 + 4i)$

e) $(1 + i)/(1 - i)$

Challenge Problems

1) Come up with a nice formula for x/y where x, y is a complex number. Hint: Look at the case where $x = 1$ first!

2) Prove the following properties of modulus (where x, y, z are complex numbers):

a) $|\overline{z}| = |z|$.

b) $|xy| = |x| \cdot |y|$.

c) $|x/y| = |x|/|y|$.

3) a) Prove that if $ax^2 + bx + c = 0$ ($a, b, c \in \mathbb{R}$) has a root z, then $\overline{z}$ is also a root.

b) Prove that if a polynomial with real coefficients (say $a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$) has root z, then $\overline{z}$ is also a root. Hint: Problem 4 will help.

c) Prove that *every* cubic polynomial has at least one real root.

4) Find all roots of the following polynomials:

a) $x^4 - 1$ b) $x^8 - 1$

For Further Investigation (Homework):

We can visualize complex number by thinking of the point (a, b) in the plane as the complex number $z = a + bi$.

Note that under this interpretation $|z|$ is simply the distance from (a, b) to the origin.

1) Try to find out what each of the following does geometrically:

a) Multiplication by i

b) Dividing by i

c) Multiplication by $1 + i$