

## COMBINATORIAL STORIES

LAMC INTERMEDIATE GROUP - 11/03/13

Last time we learned how to count! Okay, you probably already knew how to count, but we learned about  $\binom{n}{k}$ . This time we're going to prove some combinatorial identities involving  $\binom{n}{k}$ . Of course, this sounds a bit boring, so we're going to prove them by telling stories.

Here is an example (of course yours should be more creative and fun than this!) where we prove the identity you worked on last week  $\binom{15}{8} = \binom{15}{7}$ .

*Proof.* One day Derek and Pax were playing an unusual game. They had 15 distinct marbles and Morgan would hide them somewhere in the room. Then Derek and Pax would search the room until they found all the marbles. If Derek found 8 marbles, Pax must have found 7. So the number of ways for Derek to find 8 marbles is the same as the number of ways for Pax to find 7 marbles. That is  $\binom{15}{8} = \binom{15}{7}$ . □

### Here are some useful hints for all of the following problems:

- Look at each side of the equation, try to think of something one of the two sides can represent. Is there any way the other side can represent the same thing?
- Often if you have “chooses” multiplied together, it comes from making successive choices. For instance  $\binom{12}{5}\binom{5}{3}$  is like choosing 5 people from a group of 12, then choosing 3 from the group of 5.
- It is also often useful to break up a group into subgroups. For example, we can break our class into two groups: girls and boys. Then we can choose from these two subgroups.
- Finally it is also sometimes important to mark important people and count them separately. For instance, I may want to count all the ways to choose 3 of the 5 instructors. But I really care about groups with Pax in them, so I can count the groups with Pax and the groups without Pax separately.



## COMBINATORIAL IDENTITIES

Now we'll start proving identities. There's no need to do them in order so try to flip around and see if you can come up with good stories for any of the identities!

(1) To get started we will provide a skeleton story for the first problem.

$$\binom{n+m}{2} = \binom{n}{2} + \binom{m}{2} + mn$$

- (a) There are  $n$  \_\_\_\_\_ and  $m$  \_\_\_\_\_ .
- (b) We must choose 2 out of the whole group of \_\_\_\_\_ & \_\_\_\_\_. This represents the left hand side.
- (c) To do this we can:
- (i) Choose 2 from the group of \_\_\_\_\_ .
  - (ii) Choose 2 from the group of \_\_\_\_\_ .
  - (iii) or Choose 1 from each group.
- i) Represents the \_\_\_\_\_ term on the right hand side.
- ii) Represents the \_\_\_\_\_ term on the right hand side.
- iii) Represents the \_\_\_\_\_ term on the right hand side.

(2) We are also providing a story skeleton for the second problem.

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

(a) We still need to get that group of students together for a Ping-Pong table. We have  $n$  total students.

(b) The left hand side is equal to  $\binom{n}{k} \binom{k}{1}$ .

This is like choosing \_\_\_\_\_ and \_\_\_\_\_ from among them.

(c) The right hand side represents the same thing but picking in the opposite order. First we pick \_\_\_\_\_. Then we pick \_\_\_\_\_ from the remaining \_\_\_\_\_ students.

$$(3) 36! = \binom{36}{9} \cdot 9! \cdot 27!$$

$$(4) \binom{n}{2} = \frac{1}{2} \binom{n}{2} [\binom{n}{2} - 1]$$

$$(5) \binom{17}{6} \binom{6}{4} = \binom{17}{4} \binom{13}{2}$$

$$(6) \binom{81}{10} \binom{19}{0} + \binom{81}{9} \binom{19}{1} + \binom{81}{8} \binom{19}{2} + \cdots + \binom{81}{0} \binom{19}{10} = \binom{100}{10}$$

$$(7) \binom{27}{0}^2 + \binom{27}{1}^2 + \cdots + \binom{27}{27}^2 = \binom{54}{27}.$$

$$(8) \binom{10}{10} + \binom{11}{10} + \binom{12}{10} + \cdots + \binom{19}{10} = \binom{20}{10}$$

$$(9) \binom{10}{0} + \binom{11}{1} + \binom{12}{2} + \cdots + \binom{19}{9} = \binom{20}{9}$$

$$(10) \binom{100}{0} + \binom{100}{2} + \cdots + \binom{100}{100} = \binom{100}{1} + \binom{75}{3} + \cdots + \binom{100}{99}$$



$$(11) \binom{n}{0}2^n + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \cdots + \binom{n}{n}2^0 = 3^n$$

$$(12) \text{ (Pascal's triangle) } \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$