CONTINUITY PROBLEMS IN GEOMETRY (UCLA MATH CIRCLE)

The idea of using continuity is one of the oldest and ubiquitous in mathematics in general, and in geometry in particular. Below we assembled several problems where it plays a crucial role.

Warning: These problems appear simpler than they really are. When you solve them, make sure you have a complete proof, not just intuitive reasoning.

Note: Stars indicate harder problems.

1. Cutting lines

Introductory Problem 1. Let Q be a convex polygon and let L be a line in the plane $\mathbb{R}^2$. Prove that there is a parallel line $\ell \parallel L$ which divides Q into two polygons of equal area.

Introductory Problem 2. Let Q be a convex polygon and let z be a point in the plane $\mathbb{R}^2$. Prove that there is a line ℓ which goes through z and divides Q into two polygons of equal area.

Problem 3. Let Q be a convex polygon and let L be a line in the plane $\mathbb{R}^2$. Prove that there is a parallel line $\ell \parallel L$ which divides Q into two polygons of equal perimeter.

Problem 4. Let Q be a convex polygon and let z be a point in the plane $\mathbb{R}^2$. Prove that there is a line ℓ which goes through z and divides Q into two polygons of equal perimeter.

Problem 5. Let Q be a convex polygon in the plane $\mathbb{R}^2$. Prove that there is a line ℓ which divides Q into two polygons of equal area and perimeter.

Problem 6. Let Q be a convex polygon in the plane $\mathbb{R}^2$. Prove that there exist two orthogonal lines ℓ_1 and ℓ_2, which divide Q into four polygons of equal area.

Problem 7. Let Q be a convex polygon in the plane $\mathbb{R}^2$. Prove or disprove: there exist two lines ℓ_1 and ℓ_2 at angle 60°, which divide Q into four polygons of equal area.

Problem 8. Let Q_1 be a regular hexagon and Q_2 be a square in the plane $\mathbb{R}^2$. Prove that there is a line ℓ which divides each polygon into two polygons of equal area.

Problem 9. Let Q_1 and Q_2 be two convex polygons in the plane $\mathbb{R}^2$. Prove that there is a line ℓ which divides each polygons into two polygons of equal area.

Problem 10.* Let Q be a convex polygon in the plane $\mathbb{R}^2$. Prove that there exist three lines ℓ_1 and ℓ_2, which divide Q into six polygons of equal area.

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2. Cutting planes

Problem 11. Let P_1, P_2 and P_3 be three cubes in the space $\mathbb{R}^3$. Prove that there is a plane H which divides each cube into two polytopes of equal volume.

Problem 12. Let P be a tetrahedron and U be a plane in the space $\mathbb{R}^3$. Prove that there is a parallel plane $H \parallel U$ which divides P into two polytopes of equal volume.

Problem 13. Let P be a convex polytope and L be a line in the space $\mathbb{R}^3$. Prove that there is a plane H which contains L, and which divides P into two polytopes of equal volume.

Problem 14.* Let P be a convex polytope and z be a point in the space $\mathbb{R}^3$. Prove that there exist two orthogonal planes H_1 and H_2 which go though z, and which divide P into four polytopes of equal volume.

Problem 15.* Let P_1 and P_2 be two convex polytopes and z be a point in the space $\mathbb{R}^3$. Prove that there is a plane H which go through z, and which divides both polytopes into two polytopes of equal volume.

Problem 16.* Let P_1, P_2 and P_3 be three convex polytopes in the space $\mathbb{R}^3$. Prove that there is a plane H which divides each polytope P_i into two polytopes of equal volume.

Problem 17.* Let P be a convex polytope and z be a point in the space $\mathbb{R}^3$. Prove that there exist three planes H_1, H_2 and H_3 which go through z, and which divide P into six polytopes of equal volume.

Problem 18.** Let P be a convex polytope in the space $\mathbb{R}^3$. Prove that there exist three pairwise orthogonal planes H_1, H_2 and H_3 which divide P into eight polytopes of equal volume.

P.S. For more on fair division problems, see Chapter 4 in
Igor Pak, *Lectures on Discrete and Polyhedral Geometry*,
available online at `http://www.math.ucla.edu/~pak/book.htm`