METRIC SPACES ON UNUSUAL SETS

MATH CIRCLE (HS1) 10/27/2013

Recall that (M, d) a metric space if M a set and $d : M^2 \to \mathbb{R}$ such that d is a metric. d is a metric if it satisfies properties 2,3, and 4 below. We say that d is an ultrametric if d satisfies properties 2,3, and 4'.

2	d is symmetric ($d(x, y) = d(y, x)$)
3	$x = y$ if and only if $d(x, y) = 0$
4	d satisfies the triangle inequality ($d(x, z) \leq d(x, y) + d(y, z)$)
4'	$d(x, z) \leq \max\{d(x, y), d(y, z)\}$

Note that property 4' implies property 4, so any ultrametric is a metric.

In particular, recall an example of an ultrametric d (called the discrete metric) so that (S, d) is a metric space for any S:

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Today we will explore examples of metrics spaces (and non metric spaces) for more unusual sets.

Words

Let W consist of all words of length n. More formally, $W = \{x_1 x_2 \cdots x_n | x_i \in \{a, b, c, \ldots, z\}\}$.

1) (Hamming Distance) For two words $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$ in S, define

$$d(x, y) = k, \text{ where } k \text{ is the number of times } x_i \neq y_i.$$

Show that (W, d) is a metric space. Hint: Think about how the $n = 1$ case is relates to the general case.

2) For two words $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$ in S, define

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 2^{-k}, & \text{if } x \neq y \text{ and } k \text{ is the smallest integer such that } x_k \neq y_k \end{cases}.$$

Show that d is an ultrametric on W.

Points on a Circle

Suppose we have a circle $\mathcal{C}$. Let P consist of all the points on the circle $\mathcal{C}$.

3) Let, for $P, Q \in \mathcal{C}$, $d(p, q) =$ the length of the smaller arc from p to q on the circle $\mathcal{C}$. Is d a metric?

4) Let, for $P, Q \in \mathcal{C}$, $d(p, q) =$ the length of the arc going clockwise from p to q on the circle $\mathcal{C}$. Is d a metric?

On Finite Sets

Let S consist of all finite sets. Recall that if A is a set, then $|A|$ denotes the number of elements in A.

5) Let, for $A, B \in S$, $d(A, B) = ||A| - |B||$. Is (S, d) a metric space?

6) Recall that if A, B are sets, then $A \triangle B$ is the set of things in A or in B, but *not* in both. Let, for $A, B \in S$, $d(A, B) = |A \triangle B|$. Is (S, d) a metric space?

On Lines in the Plane

Let L consist of all lines in a plane.

7) Define, for ℓ_1, ℓ_2 both lines,

$$d(\ell_1, \ell_2) = \begin{cases} s, & \text{if } \ell_1, \ell_2 \text{are parallel, and } s \text{is the distance between them} \\ \theta, & \text{OW, and } \theta \in (0^\circ, 90^\circ] \text{is the angle of intersection between } \ell_1, \ell_2 \end{cases}$$

Show that d is not a metric.

Note: We can modify this to get a metric, see the homework.

For Further Investigation (Homework):

1) Suppose that (M, d) is a metric space. Prove that (M, d') is also a metric space, where

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Hint: Check that

$$d'(x, y) + d'(y, z) = \frac{d(x, y) + d(y, z) + 2 \cdot d(x, y) \cdot d(y, z)}{1 + d(x, y) + d(y, z) + d(x, y) \cdot d(y, z)}.$$

We thus need to show

$$d'(x, z) = \frac{d(x, z)}{1 + d(x, z)} \leq \frac{d(x, y) + d(y, z) + 2 \cdot d(x, y) \cdot d(y, z)}{1 + d(x, y) + d(y, z) + d(x, y) \cdot d(y, z)} = d'(x, y) + d'(y, z),$$

which isn't hard, just a bit messy!

2) As in the setup for 7), let

$$d(\ell_1, \ell_2) = \begin{cases} \frac{s}{1+s}, & \text{if } \ell_1, \ell_2 \text{ are parallel, and } s \text{is the distance between them} \\ \theta + 1, & \text{OW, and } \theta \in (0^\circ, 90^\circ] \text{ is the angle of intersection between } \ell_1, \ell_2 \end{cases}.$$

Convince yourself that d is a metric.

Hint: i) Look at cases! ii) At some point, an argument similar to that in 1) may be helpful.