Week 5 solutions!

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Problems

1. Royal Flush, Straight Flush, Four of a Kind, Full House, Flush, Straight, Three of a Kind, Two Pair, Pair, High Card

2. (a) Straight
   (b) Full House
   (c) Flush
   (d) Two Pair
   (e) High Card
   (f) Pair
   (g) Straight Flush
   (h) Four of a Kind
   (i) Royal Flush
   (j) Three of a Kind

3. (a) we have two objects, so we can put either one first and the other one second. Two ways.
   (b) There are 5 of us, so there are 5 choices for who is first. Times 4 choices for who is second, times 3 for who is third, 2 for who is fourth and the last guy is determined. Giving $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 := 5!$.
   (c) The same argument works, replacing 5 with 25.
   (d) The same argument works, replacing 25 with n.

4. (a) Lets first put a sticker on the head of one dolphin and one tiger so we can tell them apart. Then the problem with sticker heads is the same as the previous one, so we have $4 \cdot 3 \cdot 2 \cdot 1$ ways to organize these. Then we simply have to count how many of these are really the same when we peal off the stickers. To do this we have to divide by the number of ways to organize the two tigers and the number of ways to organize to the two dolphins. Giving us
   $$4 \cdot 3 \cdot 2 \cdot 1 \cdot \frac{1}{2 \cdot 2}$$
   (b) We do the same process as the last part, put stickers on there heads and organize then divide by repeats. This time we divide by 3! for the ways to arrange the three cats and 1 for the number of ways to arrange the single dog. Giving us
   $$4 \cdot 3 \cdot 2 \cdot 1 \cdot \frac{1}{3 \cdot 2 \cdot 1}$$
   (c) Use the same method as before, $n! \cdot \frac{1}{2!}$.
   (d) Use the same method as before, $n! \cdot \frac{1}{3!}$.
   (e) Use the same method as before, $n! \cdot \frac{1}{m!}$.

(f) We use the method of the first part of this problem. There are 7 letters, but there are two L's and two I's. So we slap a sticker on on L and one I and arrange them to get $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 7!$ ways of organizing the letters with the special stickers. Then divide by the number of ways to arrangements that will be the same when we remove the stickers, which in this case is $2! \cdot 2!$. So we get $\frac{7!}{2! \cdot 2!}$.

(g) This one is applying the exact method as in the word WILLIAM but replacing 7 with r, the two L by m of type A and two I by n or type B giving $\frac{r!}{n! \cdot m!}$.

5. (a) We have n choices.
   (b) we have n choices for the first ball, then $n - 1$ choices for the second ball. Giving us $n \cdot (n - 1)$
   (c) Continuing the argument gives $n(n - 1)(n - 2)$.
   (d) Continuing it further gives $n(n - 1)(n - 2) \cdots (n - k)$.

6. (a) again we have n choices of balls.
   (b) Here we have n choices for the first ball, then $n - 1$ choices for the second ball. But if we pulled say ball 1 then ball 3 this would be the same as pulling ball 3 then ball 1. So we have to divide by the number of ways to order 2 balls, which is 2. giving us $\frac{n(n-1)}{2}$.
   (c) The same argument gives us $\frac{n(n-1)(n-2)}{3!}$
   (d) The same argument gives us $\frac{n(n-1)(n-2)(n-3)\cdot(n-(k-1))}{k!}$.

7. Because choosing 7 items we want is the same as choosing 8 items we don't want.

How many of each hand is there anyway?

1. We have to choose 5 cards out of 52 cards, giving us $\binom{52}{5} = 2598960$.

2. There are only 4 Royal Flushes, since there is exactly one per suit. *(this is extremely tiny compared to the number of hands!)*

3. There are 32 Straight Flushes. First we choose a suit, giving us 4 options. Then we choose the lowest value in the straight between 2 and 9 giving us 8 choices. That is we have $4 \cdot 8$ or 32 straight flushes. *(this is extremely tiny compared to the number of hands!)*

4. There are 624 Four of a Kinds. First we choose the value between 2 and 14 to have four of, giving us 13 options. Then we have 48 extra cards we can choose from to fill up our fifth spot, giving $13 \cdot 48 = 624$.

5. There are 3744 Full-Houses. First we can choose which value appears twice, giving us 13 options. Then choose which value appears three times, giving us another 12 options, so we have $13 \cdot 12$. Then there are $\binom{4}{2}$ ways to pick out our pair and $\binom{4}{3}$ ways to pick out our triple. Giving us $13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2} = 3744$.

6. There are 5112 Flushes. We have 4 choices of suit, then $\binom{13}{5}$ ways to pick 5 values from that suit. But then 9 of these choices of 5 values yield straight or royal flushes so we have to subtract them off. Giving us $4 \cdot (\binom{13}{5} - 9) = 5112$.

7. There are 9180 Straights. First we can pick the lowest value in our straight from 2 through 10 giving us 9 possibilities. Then for each value in our straight we have 4 choices of suit giving $9 \cdot 4^5$. But there are 4 ways where we end up with every value having the same suit out of those choices. So we get $9 \cdot (4^5 - 4) = 9180$.

8. There are 54912 Three of a Kind. *This is a huge jump from the number of straights!* There are 13 values we can choose the value for our three of a kind. And the three cards from that value can be chosen in $\binom{4}{3}$ ways. Giving us $13 \cdot \binom{4}{3}$. Then we need to choose 2 extra cards, which both are distinct value and have value distinct from the value of our three of a kind. This gives us 48 options for the first card, and 44 options for the second. But, the last two cards can be picked in either order so we have to divide by the number of ways to order 2 cards. So we get $13 \cdot \binom{4}{3} \cdot 48 \cdot 44 \cdot \frac{1}{2} = 54912$.

9. There are 123552 Two Pairs. We have to choose 2 values from 13 choices, giving us $\binom{13}{2}$ then we have $\binom{4}{2}$ ways to choose two cards from each value we chose. Then we have to pick a final card, which needs to be of a different value then our two choices. So we have 44 options for the last card. Giving us $\binom{13}{2}\binom{4}{2}^2 \cdot 44 = 123552$.

10. There are 1098240 Pairs. For our pair we can choose the value in 13 ways. Then we have $\binom{4}{2}$ ways to pick the card from that value. Afterward we need to choose 3 cards, each of a different rank then our pair, giving us $48 \cdot 44 \cdot 40$ options. But again these last three can be picked in any order so we need to divide by the number of ways to order three cards, and hence we get $\binom{4}{2} \cdot 48 \cdot 44 \cdot 40 \cdot \frac{1}{3!} = 1098240$.

11. There are 1303560 High Cards. We need to choose 5 cards of different value so we have $52 \cdot 48 \cdot 44 \cdot 40 \cdot 36$ ways to do this. But they can be picked in any order so we need to divide by the number of ways to order 5 cards giving us $52 \cdot 48 \cdot 44 \cdot 40 \cdot 36 \cdot \frac{1}{5!}$. Then we need to subtract out straights, flushes, straight flushes and royal flushes. Giving us

$$52 \cdot 48 \cdot 44 \cdot 40 \cdot 36 \cdot \frac{1}{5!} - 9180 - 5112 - 32 - 4 = 1303560$$

12. Of course I guessed correctly. And yes they all add up to $\binom{52}{5}$ (plug it into your calculator!).

Extra Counting Problems

1. There are $\binom{4}{2}$ ways of being dealt 2 aces, and $4 \cdot 3$ ways of being death $2, 7$ off suit. So it looks like you're more likely to get the bad hand than the good one.

2. (a) There are 2^n. At each bit in our string we have 2 choices, and there are n bits so we have 2^n total options.

   (b) There are 2^n. We can identify subsets with strings of length n as follows: we put a 0 in the i^{th} spot if i is not in the set, and a 1 in the i^{th} spot if i is in the set. Therefore we see that we have the same number of subsets and strings of 0 and 1.

3. There are three ways. $\frac{1}{2} + \frac{1}{4} + \frac{1}{4}$, $\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$, $\frac{1}{6} + \frac{1}{3} + \frac{1}{2}$. Here is a way to prove this:

   Proof. At least one of the numbers must be bigger than $\frac{1}{3}$ because if not the sum of the three is less than 1. If one of them is $\frac{1}{2}$ then the other two have to sum to $\frac{1}{2}$. So at least one of them has to be bigger than $\frac{1}{4}$. If one of them is $\frac{1}{3}$ you get $\frac{1}{3} + \frac{1}{6}$ and if one is $\frac{1}{4}$ you get $\frac{1}{4} + \frac{1}{4}$. Giving us only these two options. So the other case is if $\frac{1}{2}$ does not appear. In this case all three numbers must be $\frac{1}{3}$. Thus there are only 3 ways to do this. □

4. We can choose one ball, giving us $\binom{2m}{1}$ or three balls giving us $\binom{2m}{3}$ etc. Giving us a total of

$$\binom{2m}{1} + \binom{2m}{3} + \cdots + \binom{2m}{2m-1}$$

   options.

5. If you know some geometry it follows from the fact that an inscribed angle is right if and only if it is supported by a diameter. So to get an obtuse triangle we must have more than half the circle between our two furthest points. That is, we must choose three on the same semicircle, which happens in 6 of the 8 possibilities.