POKER (AN INTRODUCTION TO COUNTING)

LAMC INTERMEDIATE GROUP - 10/27/13

If you want to be a succesful poker player the first thing you need to do is learn combinatorics! Today we are going to count poker hands.

A **hand** is a set of 5 cards from the deck of cards, which includes $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$ of each suit $\heartsuit, \diamondsuit, \spadesuit, \clubsuit$.

We will let the **value** of a card be the the number if it is 2-9, or 11 for Jack, 12 for Queen, 13 for King, and 14 for Ace.

Below is a list of the possible types of hands (not necessarily in order). Our goal is to list them in order of rarity and get an idea of how much better one hand is than another.

(1) **Royal-Flush:** 10 through Ace all of the same suit.
(Example: $10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit$)

(2) **Straight-Flush:** 5 cards of the same suit, with 5 values in a row. (*except* $10, J, Q, K, A$*!)*
(Example: $5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit$)

(3) **Flush:** 5 cards of the same suit. (*except if they are a straight flush!)*
(Example: $6\clubsuit, 5\clubsuit, 9\clubsuit, J\clubsuit, K\clubsuit$)

(4) **Straight:** 5 values in a row. *(except if they are a straight flush!)*
(Example: $8\clubsuit, 9\diamondsuit, 10\diamondsuit, J\heartsuit, Q\spadesuit$)

(5) **Pair:** Exactly 2 cards of the same value.
(Example: $6\clubsuit, 6\diamondsuit, 3\heartsuit, 9\diamondsuit, J\heartsuit$)

(6) **Two-Pair:** Two separate pairs.
(Example: $8\clubsuit, 8\diamondsuit, J\heartsuit, J\spadesuit, 4\heartsuit$)

(7) **Three-of-a-Kind:** Exactly three cards of the same value.
(Example: $8\diamondsuit, 8\clubsuit, 8\spadesuit, J\heartsuit, K\spadesuit$)

(8) **Four-of-a-Kind:** Four cards of the same value.
(Example: $3\diamondsuit, 3\heartsuit, 3\spadesuit, 3\clubsuit, J\diamondsuit$)

(9) **Full-House:** A Three-of-a-kind plus a pair.
(Example: $4\clubsuit, 4\spadesuit, 4\diamondsuit, 3\heartsuit, 3\clubsuit$)

(10) **High-Card:** A hand which is not of type (1)-(9).
(Example: $3\heartsuit, 9\diamondsuit, Q\heartsuit, 6\clubsuit, A\spadesuit$)

(1) Take a guess at ordering these hands in terms of rarity from most common to most rare (we'll see how good your intuition is at the end).

(2) Dustin has never played poker and is a bit confused about the types of hands. Identify the following for him:

(a) $6\clubsuit, 5\diamondsuit, 8\spadesuit, 7\heartsuit, 9\diamondsuit$:

(b) $3\clubsuit, J\heartsuit, J\diamondsuit, 3\diamondsuit, J\spadesuit$:

(c) $4\heartsuit, 9\heartsuit, J\heartsuit, K\heartsuit, Q\heartsuit$:

(d) $K\heartsuit, K\diamondsuit, 3\spadesuit, 4\clubsuit, 3\diamondsuit$:

(e) $4\clubsuit, 9\diamondsuit, 5\spadesuit, 6\heartsuit, Q\diamondsuit$:

(f) $6\heartsuit, 5\diamondsuit, 3\spadesuit, 2\clubsuit, 6\spadesuit$:

(g) $5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit$:

(h) $5\heartsuit, K\diamondsuit, K\clubsuit, K\spadesuit, K\heartsuit$:

(i) $10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit, A\clubsuit$:

(j) $J\clubsuit, 3\heartsuit, 6\spadesuit, J\spadesuit, J\diamondsuit$:

Combinations and Permutations

The single most important thing for counting our poker hands is knowing how many ways there are to choose k objects from a group of n objects. Here we'll work up to counting that by thinking about permutations and combinations.

(1) How many ways are there to order the sets with the following elements:

(a) 1 tiger and 1 lion?

(b) The Math Circle Instructors?

(c) The students in this class?

(d) n people?

(2) In the last problem we ordered things that were all distinct. What happens if we have several identical objects?

(a) How many ways are there to order 2 Tigers and 2 Dolphins? (The two tigers are indistinguishable, as are the two dolphins)

(b) How many ways are there to order 3 cats and one dog? (The three cats are indistinguishable)

(c) How many ways are there to order n objects if 2 of them are the identical?

(d) How many ways are there to order n objects if 3 of them are the identical?

(e) How many ways are there to order n objects if m of them are the identical?

(f) How many ways are there to order the letters in the name WILLIAM ?

(g) How many ways are there to order r objects if m of the objects are of type A and n of the objects are of type B?

(3) Now we want to know how many ways there are to select a subset. First, suppose that the order of picking does matter. For example, if we are picking points on the coordinate plane, picking $(2, 5)$ is much different than picking $(5, 2)$. We will call an ordered set like this a tuple.

(a) How many ways are there to choose a 1-tuple of balls from a group of n balls?

(b) How many ways are there to choose a 2-tuple of balls from a group of n balls?

(c) How many ways are there to choose a 3-tuple of balls from a group of n balls?

(d) Finally, how many ways are there to choose a k-tuple of balls from a group of n balls?

(4) Now we suppose that order doesn't matter. All we care about now is what objects we choose, not the order in which they are chosen.

(a) How many ways are there to choose 1 ball from a group of n balls?

(b) How many ways are there to choose 2 balls from a group of n balls?

(c) How many ways are there to choose 3 balls from a group of n balls?

(d) Finally, how many ways are there to choose k balls from a group of n balls?

(5) The number we calculated in part (d) of the previous problem is called "n ***choose*** k" and denoted $\binom{n}{k}$. Explain why $\binom{15}{7} = \binom{15}{8}$.

How many of each hand is there anyway?

(1) How many possible poker hands are there?

(2) How many Royal-Flushes are there?

(3) How many Straight-Flushes are there? (*Remember: A royal flush is not a straight flush!)*

(4) How many Flushes are there?

(5) How many Straights are there?

(6) How many Four-of-a-Kinds are there?

(7) How many Full Houses are there?

(8) How many Three-of-a-Kinds are there?

(9) How many High-Card are there?

(10) How many Two-Pairs are there?

(11) How many Pairs are there?

(12) To check your work, add up all of your answers from 2-11. Do they add up to the answer to number 1? (they should). Also did you guess correctly about which hand was rarest?

THE FABULOUS FIBONACCI

The Fibonacci numbers are defined as follows: $F_1 = 1, F_2 = 1$

$$F_{n+2} = F_{n+1} + F_n$$

To start listing them we can see we get $1, 1, 2, 3, 5, 8, 13, 21, 34, 55 \cdots$. Here we will use combinatorics to discover facts about these curious numbers.

Fibonacci's Chocolate Bar.

(1) In how many ways can we cover a $1 \times n$ chocolate bar with tiles (1×1) and dominoes (1×2)?

(2) Show that $F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{n-\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor}$

(3) Show that $F_n^2 + F_{n+1}^2 = F_{2n+1}$

(4) Show that $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$

(5) Calculate d_n, the n^{th} diagonal sum on Pascal's triangle. Where $d_n$is defined using the following image of pascals triangle.

$$\begin{array}{l} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \end{array}$$

Viewing this as an infinite array, we define

$$d_n = (n, 1) + (n - 1, 2) + \cdots$$

where we note that this is a finite sum with at most n.

Extra Counting Problems

(1) In a variant of poker called Texas-Hold'em each player is dealt two cards. The general knowledge is that the best hand to be dealt is two Aces, and the worst is $2, 7$ off suit (meaning the two and seven have different suits). Which hand are you more likely to be dealt? How many of each hand are there?

(2) An important method in counting is reducing one problem to another via an identification. Here is a (simple) example:

(a) How many strings of length n consisting only of 0's and 1's are there?

(b) How many subsets of the set $\{1, \cdots, n\}$ are there? (*Hint: find an identification with the first part!)*

(3) How many ways are there to find three positive integers a, b, c such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1?$$

(Note: an answer isn't sufficient, you should provide proof!)

(4) (IF YOU KNOW SOME GEOMETRY) Suppose we have three diameters of a circle, each with a pair of endpoints. How many ways can we choose one point from each pair of endpoints such that the triangle resulting from our three points is obtuse?

(5) How many ways are there to choose an odd number of balls from a collection of distinct balls of size $2m$?

HOMEWORK!

(1) Suppose instead of 5 cards we played poker with 3 cards.

  (a) How many Straight-Flushes are there?

  (b) How many Three-of-a-kind are there?

(2) In how many ways can you arrange the letters in the name VANESSA?