LAMC Beginners' Circle: Pigeonhole Principle

Problems from Handout by Oleg Gleizer
Solutions by James Newton

1. Use four fours to make thirty-nine.

Solution. $44 - \frac{\sqrt{4}}{.4} = 44 - \frac{5}{2}(2) = 44 - 5 = 39$ works. (Not a unique answer.) *Done.*

2. Is it possible to cover an equilateral triangle with two smaller equilateral triangles? Why or why not?

Solution. No. Suppose we have an equilateral triangle $\triangle A$ which is covered by two equilateral triangles, $\triangle B_1$ and $\triangle B_2$. Then every point in $\triangle A$ lies in either $\triangle B_1$ or $\triangle B_2$. In particular, the three vertices of $\triangle A$ must be distributed among the two triangles $\triangle B_1$ and $\triangle B_2$. Applying the Pigeonhole Principle, we have that either $\triangle B_1$ or $\triangle B_2$ must contain at least two of the vertices of $\triangle A$. But if this is the case, then this triangle must be at least as large as $\triangle A$. Thus it is impossible to cover an equilateral triangle with two smaller equilateral triangles. *Done.*

Note: Strictly speaking, the Pigeonhole Principle applies only if we assume that the two smaller triangles are non-overlapping. However, the logic of the proof applies even if we allow for overlapping triangles: one of $\triangle B_1$ and $\triangle B_2$ must contain at least two of the vertices of $\triangle A$. But this cannot occur without the triangle being at least as large as $\triangle A$. So the answer is still No.

3. You are given 224 integers. Prove that there exist at least two of them such that their difference is divisible by 223.

Proof. Let $a_1.a_2, \ldots, a_{224}$ be the given integers. Using long division, we can express each $a_i = 223b_i + r_i$, where $0 \leq r_i < 223$. Consider the difference $a_i - a_j$ for some $i \neq j$. $a_i - a_j = 223b_i + r_i - 223b_j - r_j = 223(b_i - b_j) + (r_i - r_j)$ is divisible by 223 if and only if $r_i - r_j = 0$, that is if $r_i = r_j$. So it suffices to show that two of the a_i have the same remainders. Consider the 224 remainders $r_1, r_2, \ldots, r_{224}$. Each of these takes only one of 223 possible values. Applying the Pigeonhole Principle, two of these remainders have the same value; that is, $r_i = r_j$ for some $i \neq j$. By our above discussion, we have found at least two integers such that their difference is divisible by 223. □

Proof 2. For students familiar with modular arithmetic. Let $a_1, a_2, \ldots, a_{224}$ be the given integers. We consider these integers modulo 223. With respect to this modulus, these 224 integers each take only one of 223 distinct values (namely, from 0 to 222); applying the Pigeonhole Principle, we conclude that there must be some $i \neq j$ such that $a_i \equiv a_j \mod 223$. Subtracting, we have $a_i - a_j \equiv 0 \mod 223$; thus $a_i - a_j$ is divisible by 223. Thus we have found two integers whose difference is divisible by 223, and so the claim is proven. □

Note: These two proofs use exactly the same steps and logic; the only difference is in language used. Using modular arithmetic is equivalent to looking at the remainders of numbers after long division.

4. **You are given a (8×8) chess-board with a pair of opposite corners cut off. You are further given a set of dominoes each equal in size to a pair of the board squares with a common side. Is it possible to tile the board with the dominoes in such a way that all the board squares are covered while the dominoes neither overlap nor stick out?**

   Solution. No. First we note that each domino will cover two adjacent squares on the chess-board. As any two adjacent squares on the board are of different colours, then each domino covers one square of each colour. Now consider the modified chess-board, with the opposite corners removed. These opposite corners will be of the same colour. Then our board has two more squares of one colour than it does of the other. But if each domino covers one of each colour, then eventually there will be two squares of this colour left over. These squares are not adjacent, and thus cannot be covered by a domino without overlaps or hanging over the edge. *Done.*

   Note: A deep consequence of this proof is the fact that we have proven that NO *solution exists. This means that if someone claims to have tiled such a chess-board in such a manner, we automatically know that something is wrong. We do not even have to see the purported solution!*

5. **The ocean covers more than half of the Earth's surface. Prove that the ocean has at least one pair of antipodal points.**

   Proof. Assume for contradiction that all points on the Earth's surface which lie over ocean have antipodes which lie over land. Then there must be at least as much land as water on the Earth's surface (by area). But we are given that the ocean covers more than half of the Earth's surface, so this cannot be. Thus our initial assumption that all points which lie over water have antipodes over land, and thus there must be at least one point which lies over ocean that has an antipode over ocean; alternatively stated, there is at least one pair of antipodal points over the ocean. □

   Better Proof. (Oleg) Consider the surface area A_O of the ocean on the planet. Consider also the surface area A_A of the antipodes of all the points which lie on the ocean. Note that $A_A = A_O$. We now look at the combined area $A_O + A_A = 2A_O$ (by equality of the areas). Because we know that the ocean covers more than half of the surface area of the planet (which we will denote by A_P), we have $A_O + A_A = 2A_O > 2(\frac{1}{2}A_P) = A_P$. Applying the geometric form of the Pigeonhole Principle, as $A_O + A_A > A_P$, and A_P represents the totality of points on the surface of the planet, we conclude that A_O and A_A must overlap. This means that there is at least one point which is both in the ocean and the antipode of a point in the ocean. Alternatively stated, there is at least one pair of antipodal points over the ocean. □

6. There are $n > 1$ people at a party. Prove that among them are at least two people who have the same number of acquaintances at the gathering. (We assume that if A knows B, then B also knows A.)

Proof. A partygoer can have anywhere from 0 to $n - 1$ acquaintances at the party. However, if someone knows nobody (and is known by nobody) then there cannot also be someone whom everybody knows (and who knows everybody else). Thus if we categorise partygoers by number of acquaintances, there cannot be people in both the 0 and $n - 1$ categories. There are then in fact $n - 1$ possible numbers of acquaintances at the party, and n partygoers. Applying the Pigeonhole Principle, there is at least one number of acquaintances which belongs to at least two people; alternatively stated, there are at least two people who have the same number of acquaintances. □

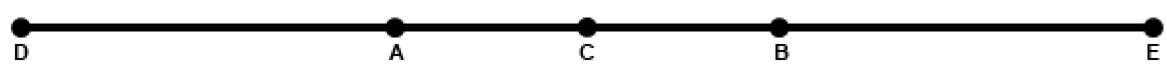
Comment: The next time you go to a party, demonstrate this for your friends! But then take care to explain to them why it must be so. If you can do this on the fly in a non-Math-Circle setting, then you truly have mastered the principle behind the proof.

7. Among any five points with integer coordinates in the plane, there exist two such that the centre of the line segment connecting them has integer coordinates as well.

Proof. Every integer is either even or odd. Thus there are four types of points with integer coordinates: (even, even), (even, odd), (odd, even) and (odd, odd). Applying the Pigeonhole Principle to the five points, there are at least two points in at least one of these categories. Call these points (x_1, y_1) and (x_2, y_2). As these points are in the same category, x_1 and x_2 are either both even or both odd, and y_1 and y_2 are either both even or both odd. Thus both $x_1 + x_2$ and $y_1 + y_2$ are even, and so $(x_1 + x_2)/2$ and $(y_1 + y_2)/2$ are integers. But $((x_1 + x_2)/2, (y_1 + y_2)/2)$ is the midpoint of the line segment between (x_1, y_1) and (x_2, y_2), proving the claim. □

8. Prove that if every point on a straight line is painted either black or white, then there exist three points of the same colour such that one is the midpoint of the line segment formed by the other two.

Proof. Let A and B be two arbitrary points of the same colour along the line. Define C to be the midpoint of segment $\overline{AB}$. Further construct points D and E such that $DA = AB = BE$. Consult the figure below for this set-up.



If C, D and E are all of the same colour, then we are done as C is the midpoint of segment $\overline{DE}$. Otherwise, there is at least one of C, D and E which is the same colour as A and B. This point then forms a triple with A and B in one of the following ways: C is the midpoint of $\overline{AB}$, A is the midpoint of $\overline{DB}$ or B is the midpoint of $\overline{AE}$. □

9. All the points in the plane are painted with one of two colours. Prove that there exist two points in the plane that have the same colour and are located exactly one foot away from each other.

Proof. (Brute-Force Method) Choose an arbitrary point P in the plane. Construct a circle C_P of radius 1 foot and centre P. If one point on C_P is the same colour as P, then we are done. Otherwise, every point on this circle has the opposite colour of P. In this case, choose another point Q on this circle. Construct another circle C_Q of radius 1 foot, but with centre Q. This circle will intersect the original circle at two points. Each of these points are then 1 foot away from Q, as they are on C_Q. However, each of these points are also on C_P, as is Q, which means that these points have the same colour as Q (the opposite colour of P). Then even in this case we have found two points one foot apart with the same colour. □

Better Proof. (Oleg) Place an equilateral triangle of side length one foot in the plane and consider the vertices. There are 3 vertices which are each coloured with one of two colours. Applying the Pigeonhole Principle, we see that at least two are of the same colour. But as they are the vertices of our equilateral triangle, they are exactly one foot apart from each other. □

Note: Both proofs are absolutely sound and thus valid arguments. We say that the second proof is better only because it is simpler and more direct. However, it may require more thought to come across on one's own. Notice also that the arguments in either proof can be generalised beyond the one foot case, to show that there are two points of the same colour located exactly a distance r apart, for ANY *distance r.*

10. Let n be an integer not divisible by 2 and 5. Show that n has a multiple consisting entirely of "1" digits.

Proof. Let $a_k := 111\ldots1$ (k digits of "1"). Then consider the numbers $a_1, a_2, \ldots, a_{n+1}$. As in Problem 3, we divide each of these by n and conclude that if any two have the same remainder, then their difference is divisible by n. Applying the Pigeonhole Principle to the $n+1$ numbers and n possible remainders (0 to $n-1$), we find that indeed there are at least two numbers with the same remainder, and thus $a_i - a_j$ is divisible by n (let $j < i$ so the difference is positive). Because of the construction of these numbers, the difference $a_i - a_j = 111\ldots1000\ldots0$, with $i-j$ digits "1" followed by i digits "0". This has the value $a_{i-j} \times 10^i$. As n is not divisible by 2 or 5, it is not divisible by 10, and the 10^i factor in $a_i - a_j$ is not responsible for why $a_i - a_j$ is divisible by n. It follows that a_{i-j} is divisible by (equivalently, is a multiple of) n, which by construction consists entirely of digits "1". □

11. **Prove that for any $n > 1$, there exists an integer made of only "7" and "0" digits that is divisible by n.**

Proof. The proof follows a similar structure as in Problem 10. Let $a_k := 777\ldots7$ (k digits of "7"). Then consider the numbers $a_1, a_2, \ldots, a_{n+1}$. Applying the Pigeonhole Principle to the $n+1$ numbers and n possible remainders (0 to $n-1$), we find that there are at least two numbers with the same remainder, and thus (as explained in Problem 3) $a_i - a_j$ is divisible by n (again, we let $j < i$ so the difference is positive). Because of the construction of these numbers, the difference $a_i - a_j = 777\ldots7000\ldots0$, with $i-j$ digits "7" followed by i digits "0". And so we have constructed a number made only of digits "7" and "0" which is divisible by n. □

12. **Let n be an odd number. Let $a_1, a_2, \ldots, a_n$ be a permutation of the numbers $1, 2, \ldots, n$. Prove that the product $(a_1 - 1) \times (a_2 - 2) \times \cdots \times (a_n - n)$ is even.**

Proof. The final product is composed of the factors $(a_i - i)$. If even one of these factors is even, then the entire product is even. As n is odd, there are $(n+1)/2$ odd numbers in $\{1, 2, \ldots, n\}$ and $(n-1)/2$ (one less) even numbers in the set. We try to pair off odd numbers (say, as a_i) with even numbers (say, as i) so that the difference of these numbers (say, $a_i - i$) is odd. But there is one more odd number than even numbers. Thus it will have to be paired with another odd number, and the corresponding factor will be even. Thus the product $(a_1 - 1)(a_2 - 2)\cdots(a_n - n)$ is even. □

13. **A stressed-out UCLA student consumes at least one espresso every day of a particular year, drinking 500 overall. Prove that on some consecutive sequence of whole days the student drinks exactly 100 espressos.**

Proof. Let a_k denote the cumulative number of espressos consumed by the end of the kth day. For instance, $a_1 \geq 1$ and $a_{365} = 500$. In particular, as the student consumes at least one espresso every day, we have $1 \leq a_1 < a_2 < \cdots < a_{364} < a_{365} = 500$. Now also consider the values $a_k + 100$, which are also distinct as the a_k are distinct. Note that $a_{365} + 100 = 500 + 100 = 600$. Now we have $365 \times 2 = 730$ values $a_1, a_2, \ldots, a_{365}, a_1 + 100, a_2 + 100, \ldots, a_{365} + 100$, all of which represent values from 1 to 600. Applying the Pigeonhole Principle, there are at least two of our 730 numbers which have the same value. As the a_k are distinct and the $a_k + 100$ are distinct, we can furthermore conclude that $a_j = a_i + 100$ for some $i < j$. Then in the sequence of days beginning with the $(i+1)$th day and ending with the jth day, the student consumed exactly 100 espressos. □

Note: Again, this proof can be adapted to show that there is a set of consecutive days during which the student consumed exactly n espressos, for n ranging from 1 to 229, inclusive. Why must we set this maximum limit for n?

14. Prove that at a party with ten or more people, there are either three mutual acquaintances or four mutual strangers.

Proof. Select an arbitrary person P at the party. First we suppose that P has at least four friends. If there is a pair of mutual acquaintances among this group, then this pair taken along with P forms a group of three mutual acquaintances. Otherwise, any four of these friends of P form a group of four mutual strangers. If P does not have at least four friends, then P has less than 4 friends, or at least 7 strangers. Select one of these strangers, Q. Suppose Q has at least 3 acquaintances. If these acquaintances of Q are mutual strangers to each other, then taken with P we have a group of four mutual strangers. Otherwise, there at least two acquaintances of Q who know each other, and taken with Q form a group of three mutual acquaintances. If Q does not have at least 3 acquaintances, then there are at least 3 strangers to Q. If these strangers are mutual acquaintances to each other, then there is a group of three mutual acquaintances. Otherwise, there are at least two mutual strangers. These are mutual strangers both to Q and P, and so we have a group of four mutual strangers. This covers every case, and in each we have found either a group of three mutual acquaintances or four mutual strangers, thus proving the claim. □

BEYOND THE PROBLEM: *Consider a set of points called* nodes, *with each node connected to every other node. These connections are called* edges, *and the resulting figure is called a* complete graph. *Suppose we fix finitely many distinct colours, and colour every edge of a complete graph one of these colours. Then a marvelous result called* Ramsey's Theorem *tells us that if we have enough nodes in our complete graph, there is a "monochromatic complete subgraph".*

This means that we can choose some subset of our nodes to form another complete subgraph such that all of the edges are of the same colour. An equivalent statement of Problem 14 in the language of these graphs would be "Given a complete graph of 10 nodes, with each edge coloured one of two colours, show that there is a 3-node complete subgraph of one colour, or a 4-node subgraph of the other colour." In this case, one colour would represent "acquaintances" and the other colour would represent "strangers". The complete subgraphs would represent either mutual acquaintances or mutual strangers.

Unfortunately, while Ramsey's Theorem tells us that we can find some *one-colour subgraph in this problem, it does not tell us* how *to find* these *subgraphs. So we still have to go case-by-case to show that whatever the social structure of the party, we can find either three mutual friends or four mutual strangers.*

15. **Given a table with a marked point O and with 2013 properly working watches put down on the table, prove that there exists a moment in time when the sum of the distances from O to the watches' centres is less than the sum of the distances from O to the tips of the watches' minute hands.**

Proof. We divide the face of each watch into two halves in the following manner: for each watch, draw a line from the point O to the centre of the watch. Now construct the diameter of the watch face which is perpendicular to this line. One half is closer to O than the other. In particular, if the minute hand of the watch lies in the closer half, then the distance from O to the minute hand is less than the distance from O to the centre of the watch. It suffices to show that there is some point in time when more than half of the watches have minute hands lying in the distant half. We define two regions: one is the union of the "closer halves" of each watch, and the other is the union of the "distant halves" of each watch. Fix some point in time. In distributing the minute hands of the 2013 watches among these two regions, applying the Pigeonhole Principle we see that one region must have at least 1007 minute hands, which is more than half of the minute hands. If this is the distant region, then we are done. Otherwise, this is the closer region. In this case, wait half an hour when the minute hand of each watch lies in the other half. Now there is a majority of minute hands in the distant half. □

Proof 2. (Oleg) Number the watches $1, 2, \ldots, 2013$ and consider the vectors $\mathbf{OM}_i(t)$ (from O to the tip of the minute hand) and $\mathbf{OC}_i$ (from O to the centre) for watch i. Note that $\mathbf{OM}_i(t)$ is periodic with period 60 minutes, $\mathbf{OC}_i$ is constant with respect to time and $\mathbf{OM}_i(t) + \mathbf{OM}_i(t+30) = 2 \cdot \mathbf{OC}_i$ (vector sum). Summing over all the watches, we have $\sum_{i=1}^{2013} \mathbf{OC}_i = \frac{1}{2} \sum_{i=1}^{2013} (\mathbf{OM}_i(t) + \mathbf{OM}_i(t+30))$. (From now on, we omit the limits on the sum.) Applying the Triangle Inequality over a normed vector space, we have $\sum |\mathbf{OC}_i| \leq \frac{1}{2} \sum |\mathbf{OM}_i(t) + \mathbf{OM}_i(t+30))|$. However, as $|\mathbf{OM}_i(t)|$ is periodic, continuous and nonnegative, there must be some time τ within an hour where $\sum(\mathbf{OM}_i(\tau)) = \sum(\mathbf{OM}_i(\tau + 30))$. At this time, then, $\sum |\mathbf{OC}_i| \leq \frac{1}{2} \sum |2 \cdot \mathbf{OM}_i(\tau)| = \sum |\mathbf{OM}_i(\tau)|$. In words, the sum of the distances to the watch centres is less than the sum of the distances to the watch minute hands at time τ. □

Note: If you really think about it, the logic behind the two proofs is somewhat the same, in the sense that both divide the hour into two regions. But the latter proof requires an understanding of vector arithmetic, metric space topology and analysis, which I am sure you will all study to some degree at some point, but which may be a bit over-complicated at the moment. The first proof, however, uses the Pigeonhole Principle more explicitly.

Closing Remarks: Unlike previous handouts, *every single problem* in this handout asked you to prove some statement. To do so, you had to think very carefully about the problem, decide what the important points to show were, decide the best course of action to show those points, and then go on and explain your logic with precision and without relying on specific examples to show a general truth. If you made it through the entire handout, congratulations! I hope you appreciate the power of proofs as you further your mathematical education. Remember that practice makes ... maybe not *perfect*, but better.

— J.N