

# UCLA Math Circle: Quaternions

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- **Motivations and definitions:**

A complex numbers  $a + bi$  can be considered as a pair of real numbers  $(a, b)$ . One could add and multiply pairs like  $(a, b)$  and  $(c, d)$  such that the distributive property also holds. This is then known as an *algebra* over  $\mathbb{R}$ . Furthermore, the operations are commutative, associative and multiplicative inverses exist for every nonzero complex number. So one could define division as multiplying the multiplicative inverse. Set of numbers satisfying these properties is call a *commutative, division algebra*.

**Definition 1.** *The set  $\mathbb{R}^n$  with a multiplication*

$$m : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$(x, y) \mapsto m(x, y)$$

*is said to be an  $\mathbb{R}$ -algebra, or real algebra, if the multiplication the distribution laws holds, i.e.*

$$m((\alpha x + \beta y), z) = \alpha m(x, z) + \beta m(y, z),$$
$$m(x, (\alpha y + \beta z)) = \alpha m(x, y) + \beta m(x, z),$$

*for any  $\alpha, \beta \in \mathbb{R}, x, y, z \in \mathbb{R}^n$ .*

**Definition 2.** *An algebra  $\mathcal{A}$  is a division algebra if given  $a, b \in \mathcal{A}$  with  $a \neq 0$ , the two equations  $aX = b$  and  $XA = b$  have unique solutions in  $\mathcal{A}$ .*

*Example:*

1.  $\mathbb{R}$  and  $\mathbb{C}$  are both real division algebras.
2.  $\text{Mat}(2, \mathbb{R})$ , the set of  $2 \times 2$  matrices with real entries is an  $\mathbb{R}$ -algebra, but not a division algebra. Similarly,  $\text{Mat}(2, \mathbb{C})$ , the set of  $2 \times 2$  matrices with complex entries is an  $\mathbb{R}$ -algebra.

- **A Model of the Quaternions**

For  $\mathbb{R}^4$ , we could define a multiplication as follows. Let

$$e = e_1 = (1, 0, 0, 0),$$
$$i = e_2 = (0, 1, 0, 0),$$
$$j = e_3 = (0, 0, 1, 0),$$
$$k = e_4 = (0, 0, 0, 1).$$

Then every element in  $\mathbb{R}^4$  can be written as  $\alpha e + \beta i + \gamma j + \delta k$  for unique  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ . Define the multiplication by

$$i^2 = j^2 = k^2 = ijk = -e. \quad (1)$$

The set  $\mathbb{R}^4$  with this operation is called the *quaternions*, denoted by  $\mathbb{H}$ .

*Exercise:* Derive the following commutation relationships from equation (1) above.

$$\begin{aligned} ij &= -ji, \\ jk &= -kj, \\ ki &= -ik. \end{aligned}$$

*Exercise:* Find the product

$$(\alpha e + \beta i + \gamma j + \delta k)(\alpha' e + \beta' i + \gamma' j + \delta' k).$$

*Exercise:* Verify that for any  $x = \alpha e + \beta i + \gamma j + \delta k$ , we have

$$x^2 = 2\alpha x - (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)e.$$

#### • Another Model of $\mathbb{H}$

There is a map between  $\mathbb{R}$ -algebras

$$\begin{aligned} \Phi : \mathbb{C} &\longrightarrow \text{Mat}(2, \mathbb{R}), \\ a + bi &\mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \end{aligned}$$

The image of  $\Phi$  is a *subalgebra* of  $\text{Mat}(2, \mathbb{R})$ . The following exercise shows that the subalgebra of  $\text{Mat}(2, \mathbb{R})$  behaves the same way as  $\mathbb{C}$ . In this case, we say that the map is a homomorphism.

*Exercise:* Verify that  $\Phi$  satisfies the following properties

1.  $\Phi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .
2.  $c\Phi(a + bi) = \Phi(c(a + bi))$ .
3.  $\Phi(a + bi) + \Phi(c + di) = \Phi((a + bi) + (c + di))$ .
4.  $\Phi(a + bi)\Phi(c + di) = \Phi((a + bi)(c + di))$ .

Similarly, there is a homomorphism between  $\mathbb{H}$  and a subalgebra of  $\text{Mat}(2, \mathbb{C})$ . Let

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

The map is described as follows.

$$\begin{aligned} \Phi : \mathbb{H} &\longrightarrow \text{Mat}(2, \mathbb{C}), \\ (\alpha, \beta, \gamma, \delta) &\mapsto \begin{pmatrix} \alpha + \beta i & -\gamma - \delta i \\ \gamma - \delta i & \alpha - \beta i \end{pmatrix} = \alpha E + \beta I + \gamma J + \delta K. \end{aligned} \quad (2)$$

As we saw earlier, it is easier to compute with quaternions than with  $2 \times 2$  matrices. The set  $\{E, -E, I, -I, J, -J, K, -K\}$  is a group under multiplication. It is the (finite) quaternion group. Cayley was familiar with representing quaternions using 2 by 2 matrices (1858).

Let  $\text{SU}(2) = \{A \in \text{Mat}(2, \mathbb{C}) \mid A\bar{A}^t = E, \det(A) = 1\}$ . Then we have

$$\text{SU}(2) = \left\{ \begin{pmatrix} w & -z \\ \bar{z} & \bar{w} \end{pmatrix} \in \text{Mat}(2, \mathbb{C}) : |w|^2 + |z|^2 = 1 \right\}.$$

There is an isomorphism between  $\text{SU}(2)$  and the unit quaternions from the algebra homomorphism (2) above.

- **$\mathbb{H}$  as a Euclidean Space**

Let  $x = \alpha e + \beta i + \gamma j + \delta k$ ,  $y$  similarly defined. Then we can define conjugate of  $x$  to be

$$\bar{x} = \alpha e - \beta i - \gamma j - \delta k. \quad (3)$$

With respect to multiplication, we have

$$\overline{xy} = \bar{y}\bar{x}. \quad (4)$$

Then we can define the magnitude of  $x \in \mathbb{H}$  to be

$$|x|^2 = x\bar{x}.$$

If a quaternion has magnitude 1, then we call it a *unit quaternion*. Notice we have the “four square theorem”:

$$|wz|^2 = (wz)\overline{(wz)} = wz\bar{z}\bar{w} = w|z|^2\bar{w} = |w|^2|z|^2.$$

Similar to the complex numbers, let

$$\operatorname{Re}(x) = \alpha, \operatorname{Im}(x) = \beta i + \gamma j + \delta k.$$

From that, we have

$$\operatorname{Re}(x) = \frac{x + \bar{x}}{2}, \operatorname{Im}x = \frac{x - \bar{x}}{2}. \quad (5)$$

Then we can define a scalar product on  $\mathbb{H}$  as

$$\langle x, y \rangle := \alpha\alpha' + \beta\beta' + \gamma\gamma' + \delta\delta'. \quad (6)$$

*Exercise:*

1. Find  $|i|^2, \langle i, j \rangle, \langle i, k \rangle$  etc.
2. Prove the orthogonality criterion:

$$\langle x, y \rangle = 0 \Leftrightarrow x\bar{y} = -y\bar{x}.$$

3. Show that the inner product can be expressed in terms of conjugates,

$$\langle x, y \rangle = \operatorname{Re}(x\bar{y}) = \frac{x\bar{y} + y\bar{x}}{2}.$$

4. Prove that  $\langle x, y \rangle$  is symmetric and  $\mathbb{R}$ -linear

$$\langle x, y \rangle = \langle y, x \rangle.$$

5. Verify that

$$|x|^2 = x\bar{x} = \operatorname{Re}(x\bar{x}) = \langle x, x \rangle = \alpha^2 + \beta^2 + \gamma^2 + \delta^2.$$

• **Imaginary space of  $\mathbb{H}$**

The imaginary space of  $\mathbb{H}$  is defined to be

$$\operatorname{Im}\mathbb{H} := \{bi + cj + dk : b, c, d \in \mathbb{R}\}.$$

By previous exercise,  $x \in \operatorname{Im}\mathbb{H}$  if and only if  $x^2 < 0$ .

*Exercise:* Show that for purely imaginary quaternions  $u, v$ , we have  $uv + vu \in \mathbb{R}$ .

Given purely imaginary quaternions  $u, v$ , then we have

$$uv = -\langle u, v \rangle e + u \times v, \quad (7)$$

where  $u \times v$  is the vector cross product. From this and  $u \times v = -v \times u$ , we have

$$u \times v = \frac{1}{2}(uv - vu), \langle u, v \rangle e = -\frac{1}{2}(uv + vu) \forall u, v \in \text{Im}\mathbb{H}.$$

*Exercise:* Show that every nonzero quaternion  $x \in \mathbb{H}$  can be represented in infinitely many different ways as the product  $x = uv$  of two purely imaginary quaternions  $u, v$

*Exercises:* Derive the following formula

1.  $u \times (v \times w) = \frac{1}{2}(uvw - vwu)$ .
2. (Grassman identity):  $u \times (v \times w) = \langle u, w \rangle v - \langle u, v \rangle w$ .
3. (Jacobi identity):  $u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0$ .

By taking the magnitudes of both sides of equation (7), we have

$$\langle u, v \rangle^2 + |u \times v|^2 = |u|^2 |v|^2.$$

If we write  $\langle u, v \rangle = |u||v| \cos \varphi$ , then  $|u \times v| = |u||v| \sin \varphi$ .

- **Spatial Rotation** The imaginary quaternions can be identified with  $\mathbb{R}^3$  and the quaternions are closely related to the rotation of three space.

**Theorem** Let  $q = a + bi + cj + dk$  be a unit quaternion. We can write it as

$$q = \cos(\alpha/2) + u \sin(\alpha/2),$$

where  $u \in \text{Im}\mathbb{H}$  and has magnitude 1. Then the following map is the rotation of  $\mathbb{R}^3$  with respect to the axis  $u$  by a degree of  $\alpha$  in the counterclockwise direction

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}^3, v \mapsto qvq^{-1}.$$

*Proof.* Write  $q = \cos(\alpha/2) + u \sin(\alpha/2)$ . Then

$$\begin{aligned} qvq^{-1} &= (\cos(\alpha/2) + u \sin(\alpha/2))v(\cos(\alpha/2) - u \sin(\alpha/2)) \\ &= \cos^2(\alpha/2)v + \sin(\alpha/2) \cos(\alpha/2)(uv - vu) - \sin^2(\alpha/2)uvu \\ &= \cos(\alpha)v + \sin^2(\alpha/2)v + \sin(\alpha)u \times v - \sin^2(\alpha/2)uvu \\ &= \cos(\alpha)v + \sin(\alpha)u \times v + \sin^2(\alpha/2)(v - uvu) \\ &= \cos(\alpha)v + \sin(\alpha)u \times v + \sin^2(\alpha/2)2\langle u, v \rangle u \\ &= \cos(\alpha)v + \sin(\alpha)u \times v + (1 - \cos(\alpha))\langle u, v \rangle u. \end{aligned}$$

Now if  $\langle v, u \rangle = 0$  and  $\langle v, v \rangle = 1$ , then

$$\begin{aligned}\langle qvq^{-1}, v \rangle &= \cos(\alpha)\langle v, v \rangle + \sin(\alpha)\langle (u \times v), v \rangle + 0 \\ &= \cos(\alpha).\end{aligned}$$

Similarly,

$$\langle qvq^{-1}, u \times v \rangle = \sin(\alpha).$$

So  $qvq^{-1}$  is a rotation around axis  $u$  with angle  $\alpha$  in the counterclockwise direction.  $\square$

**Corollary:** Compositions of rotations are still rotations.