Bugs on a Log

1. 2013 bugs are on a stalk of celery which is one metre long. Each begins walking left or right on the celery, but if two bugs meet, both of them change directions and go on without loss of speed. What is the longest time it could take for all the bugs to walk off the end of the celery?

2. At 2013 distinct points of a circular bug race course, 2013 bugs are ready to start crawling. Each of them crawls the course in an hour. At a given signal every bug selects one of the two possible directions and starts immediately. Whenever two bugs meet both of them change directions and go on without loss of speed.[1]

   (a) Show that at a certain moment, every bug will be at its respective starting point (all at the same time).

   (b) Describe an arrangement of bugs so that it takes as long as possible for the above to happen.

3.* Let m be a positive integer and consider an m by m chequered board whose squares have sidelength 1. At the centres of some of these unit squares there is an bug. At time 0 each bug starts moving with speed 1 parallel to some edge of the chequered board. When two bugs moving in opposite directions meet, they both turn 90 degrees clockwise and continue moving with speed 1. When more than two bugs meet, or when two bugs moving in perpendicular directions meet, the bugs continue moving in the same direction as before they met. When an bug reaches one of the edges of the chequered board, it falls off and will not re-appear.[2]

   (a) Do the bugs necessarily fall off, or is it possible that some of them keep crawling around the board forever?

   (b) If it is posssible that they keep crawling forever, describe how to arrange them so that this happens. If it is not possible, what is the longest it might take for the last bug to fall off the board?

[1] Mathematical Miniatures by Savchev and Andreescu
[2] 2011 IMO Shortlist

Tiling Problems

3. Consider a square table of side length 1 metre. Determine whether it is possible to cover this table with ten square napkins of area 1/5 square metres such that every point on the table is covered twice. (Napkins can be folded.)

4. Let $R_1, R_2, \ldots$ be a sequence of rectangles such that the area of R_n is n^2. Determine whether it is always possible, no matter what the dimensions of the rectangles are, to arrange them so that they cover the plane. If so, prove it, or if it is not always possible, give an example to show why.

6.* Let a, b, M be positive integers and let $ABCD$ be a square with side length M. $ABCD$ is a union of almost disjoint rectangles, each of which has side lengths a, b. Let R be the set of rectangles intersecting AC. Prove that AC halves the area of R. (A union of rectangles is said to be almost disjoint if the interiors of the rectangles are disjoint.)