LAMC Beginners' Circle **October 13, 2013**

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Solution key

Warm-up

The following problem was communicated to me by one of our students, Arul Kolla.

Problem 1 *Use four fours to make fifty.*

$$44 + \frac{4!}{4} = 4! + 4! + \frac{4}{\sqrt{4}} = 50$$

Note 1 *Using four fours, you can make any number from zero to 50. Try it at home!*

Problem 2 *Ten numbers are written in a line. The first number equals seven. The sum of any three consecutive numbers equals fifteen. What is the last number?*

Let us call the numbers $a_1, a_2, \ldots, a_{10}$. We know that $a_1 = 7$, hence $a_2 + a_3 = 8$. Since $a_2 + a_3 + a_4 = 15$, we see that $a_4 = 7$. Similar reasoning gives $a_7 = a_{10} = 7$.

Back to place-value numerals

Example 1 *Represent* 46_{10} *as a hexadecimal.*

In the decimal system, $46 \div 16 = 2$ *rem* 14. *Hence, the following is true.*

$$46_{10} = 2_{10} \times 16_{10} + 14_{10} = 2e_{16}$$

Problem 3 *Represent* 31_{10} *as a hexadecimal.*

$$31_{10} = 16_{10} + 15_{10} = 1f_{16}$$

Example 2 *Represent* 571_{10} *as a hexadecimal.*

In the decimal system, $571 \div 16 = 35$ *rem* 11. *In other words,* $571 = 16 \times 35 + 11$. *Now,* $35 > 16$, *so we need to continue. The formula* $35 = 2 \times 16 + 3$ *leads to the following.*

$$571 = 16 \times (2 \times 16 + 3) + 11 = 2 \times 16^2 + 3 \times 16 + 11$$

So, here is the answer.

$$571_{10} = 23b_{16}$$

Problem 4 *Represent* 967_{10} *as a hexadecimal.*

$$967 \div 16 = 60 \ r \ 7$$

$$60 \div 16 = 3 \ r \ 12$$

$$967 = 60 \times 16 + 7 = (3 \times 16 + 12) \times 16 + 7$$

$$967 = 3 \times 16^2 + 12 \times 16 + 7 = 3c7_{16}$$

Problem 5 *Represent* $3,258_{10}$ *as a hexadecimal.*

Solving as above gives the following.

$$3,258_{10} = cba_{16}$$

Problem 6 *Represent the trinary number* 12121_3 *as a hexadecimal.*

$$12121_3 = 3^4 + 2 \times 3^3 + 3^2 + 2 \times 3 + 1 = 81 + 54 + 9 + 6 + 1 = 151_{10}$$

$$151_{10} = 97_{16}$$

Problem 7 *Represent* 176_{10} *as an octal number.*

$$176 \div 8 = 22$$

$$22 \div 8 = 2 \ r \ 6$$

$$176 = 22 \times 8 = (2 \times 8 + 6) \times 8$$

$$176 = 2 \times 8^2 + 6 \times 8^1 + 0 \times 8^0 = 260_8$$

Problem 8 *What is the number* 10_b *in any base* b*?*

$$10_b = b$$

Problem 9 *What is the number* 101_b *in any base* b*?*

$$101_b = b^2 + 1$$

Problem 10 *Oleg looked at the sheet of paper Anton was working with and noticed the following computation.*

$$13^2 = 171$$

"This cannot be true!" said Oleg. "The decimal system is boring," responded Anton, "I computed this square in a base different from ten." What was the base of the place-value system Anton used for the above computation?

Recall the following identity.

$$(x+y)^2 = x^2 + 2xy + y^2 \tag{1}$$

In some unknown base, $13_b = b + 3$*. According to 1, we get the following.*

$$13_b^2 = (b+3)^2 = b^2 + 6b + 9 \tag{2}$$

On the other hand,

$$171_b = b^2 + 7b + 1 \tag{3}$$

Comparing 2 and 3 yields

$$b = 8$$

Problem 11 *Use Egyptian multiplication (and, if needed, the table on page 9 of the first handout) to compute the following product.*

𓍢 ∩ ∩ ∩ | | | | | | | |	*times*	𓍢 𓍢 ∩ ∩ ∩ ∩ ∩ | | |

138	*253*
1	*253*
2	*506*
4	*1,012*
8	*2,024*
16	*4,048*
32	*8,096*
64	*16,192*
128	*32,384*

$138 = 128 + 8 + 2$, *hence* $138 \times 253 = 32,384 + 2,024 + 506 = 34,914$. *In the original notations,*

𓍢 ∩ ∩ ∩ | | | | | | | |	*times*	𓍢 𓍢 ∩ ∩ ∩ ∩ ∩ | | |	*equals*	𓂭 𓂭 𓂭 𓆼 𓆼 𓆼 𓆼 𓍢 𓍢 𓍢 𓍢 𓍢 𓍢 𓍢 𓍢 𓍢 ∩ | | | |

Problem 12 *Perform the following long subtraction of the trinary numbers without switching to the decimals.*

$$\begin{array}{rcccccc} & 1 & 2 & 0 & 1 & 2 & 1 \\ - & & 1 & 2 & 2 & 1 & 2 \\ \hline & 1 & 0 & 0 & 2 & 0 & 2 \end{array}$$

Then convert all the three numbers to the decimal form and check your answer.

$$120121_3 = 3^5 + 2 \times 3^4 + 3^2 + 2 \times 3 + 1 = 421_{10}$$

$$12212_3 = 3^4 + 2 \times 3^3 + 2 \times 3^2 + 3 + 2 = 158_{10}$$

$$100202_3 = 3^5 + 2 \times 3^2 + 2 = 263_{10}$$

$$421_{10} - 158_{10} = 263_{10}$$

Problem 13 *Perform the following long addition of the hexadecimal numbers without switching to the decimals.*

$$\begin{array}{cccc} & a & b & c \\ + & d & e & f \\ \hline 1 & 8 & a & b \end{array}$$

Then convert all the three numbers to the decimal form and check your answer.

$$abc_{16} = 10 \times 16^2 + 11 \times 16 + 12 = 2,748_{10}$$

$$def_{16} = 13 \times 16^2 + 14 \times 16 + 15 = 3,567_{10}$$

$$18ab_{16} = 16^3 + 8 \times 16^2 + 10 \times 16 + 11 = 6,315_{10}$$

$$2,748_{10} + 3,567_{10} = 6,315_{10}$$

Problem 14 *Complete the binary multiplication table below. Use it to perform the following long multiplication without switching to decimals.*

$\times$	0	1
0	0	0
1	0	1

$$\begin{array}{cccccccc} & & & & 1 & 0 & 1 & 1 \\ & & \times & & & 1 & 1 & 0 \\ \hline & & & & 0 & 0 & 0 & 0 \\ + & & & 1 & 0 & 1 & 1 & \\ & & 1 & 0 & 1 & 1 & & \\ \hline & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Then convert both factors and the product to the decimal form and check your answer.

$$1011_2 = 8 + 2 + 1 = 11_{10}$$

$$110_2 = 4 + 2 = 6_{10}$$

$$1000010_2 = 64 + 2 = 66$$

$$11_{10} \times 6_{10} = 66_{10}$$

Problem 15 *Complete the trinary multiplication table below. Use it to perform the following long multiplication without switching to decimals.*

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	11

$$\begin{array}{ccccccc}
 & & & 1 & 2 & 0 & 2 \\
\times & & & & 1 & 1 & 2 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 \\
+ & & 1 & 2 & 0 & 2 & \\
 & 1 & 2 & 0 & 2 & & \\
\hline
 & 2 & 2 & 0 & 1 & 0 & 1
\end{array}$$

Then convert both factors and the product to the decimal form and check your answer.

$$1202_3 = 27 + 18 + 2 = 47_{10}$$

$$112_3 = 9 + 3 + 2 = 14_{10}$$

$$220101_3 = 2 \times 3^5 + 2 \times 3^4 + 9 + 1 = 658_{10}$$

$$47_{10} \times 14_{10} = 658_{10}$$

Problem 16 *Complete the octal multiplication table below.*

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4	0	4	10	14	20	24	30	34
5	0	5	12	17	24	31	36	43
6	0	6	14	22	30	36	44	52
7	0	7	16	25	34	43	52	61

Use the table to perform the following long multiplication of octal numbers without switching to the decimals.

$$
\begin{array}{ccccc}
 & & & 7 & 6 \\
\times & & & 4 & 5 \\
\hline
 & & 4 & 6 & 6 \\
+ & 3 & 7 & 0 & \\
\hline
 & 4 & 3 & 6 & 6
\end{array}
$$

Find the decimal representations of the factors and of the product and check the answer.

$$76_8 = 56 + 6 = 62_{10}$$

$$45_8 = 32 + 5 = 37_{10}$$

$$4,366_8 = 4 \times 8^3 + 3 \times 8^2 + 6 \times 8 + 6 = 2,294_{10}$$

$$62_{10} \times 37_{10} = 2,294_{10}$$

Problem 17 *Use the hexadecimal multiplication table at the end of the second handout to perform the following long multiplication without switching to the decimals.*

$$\begin{array}{rcccc} & & & d & 3 \\ \times & & & 7 & c \\ \hline & & 9 & e & 4 \\ + & 5 & c & 5 & \\ \hline & 6 & 6 & 3 & 4 \end{array}$$

Then convert the factors and the product to the decimal form and check your answer.

$$d3 = 13 \times 16 + 3 = 211_{10}$$

$$7c = 7 \times 16 + 12 = 124_{10}$$

$$6,634_{16} = 6 \times 16^3 + 6 \times 16^2 + 3 \times 16 + 4 = 26,164_{10}$$

$$211_{10} \times 124_{10} = 26,164_{10}$$

Problem 18 *Use long division to solve the following problem (with the remainder) without switching to decimals.*

$$1101101_2 \div 11_2 = 100100_2 \ r\ 1$$

There are two ways to check whether your solution of Problem 18 is correct.

Problem 19 *Multiply the quotient from Problem 18 by the denominator and add the remainder. See if you get back the numerator this way. Do not switch to decimals.*

$$100100_2 \times 11_2 + 1 = 1101101_2$$

Problem 20 *Find the decimal representations of all the relevant numbers from Problem 18 and check the correctness of its solution by performing long division in the decimal system.*

$$1101101_2 = 64 + 32 + 8 + 4 + 1 = 109_{10}$$

$$100100_2 = 32 + 4 = 36_{10}$$

$$11_2 = 2 + 1 = 3_{10}$$

$$109_{10} = 36_{10} \times 3_{10} + 1$$

From hexadecimals to binaries and back again

Sixteen is a power of two. That is why there exists a very fast and efficient way of switching back and forth between the binaries and hexadecimals that does not involve the decimals.

Example 3 *Represent the number $f3$ in the binary form. (Note that we do not need the base subscript for this number – the digit f clearly indicates hexadecimals.)*

Let us take the binary equivalent of f from the conversion table on page 15 of the first handout.

$$f = 1111_2$$

The binary equivalent of three taken from the same table is $3 = 11_2$. Let us add two zeros in front of the ones to form a group of four digits.

$$3 = 0011_2$$

(The procedure is called padding.) Finally, let us join the groups of four bits (a bit is a short form for "binary digit") together in the proper order.

$$f3 = 1111,0011_2$$

Problem 21 *Convert both numbers to decimals to check whether the above equality is correct.*

$$f3_{16} = 15 \times 16 + 3 = 243_{10}$$

$$1111,0011_2 = 128 + 64 + 32 + 16 + 2 + 1 = 243_{10}$$

The trick works as well in the opposite direction.

Example 4 *Convert the number* 1010011_2 *to the hexadecimal form.*

Let us split the above number into groups of four bits using padding if needed.

$$1010011 = 0101,0011$$

According to the conversion table, $0101_2 = 101_2 = 5_{16}$ *and* $0011_2 = 11_2 = 3_{16}$. *Writing the above hexadecimal digits in the proper order finishes the solution.*

$$1010011_2 = 53_{16}$$

Problem 22 *To check whether the above equality is correct, convert both numbers to the decimal form.*

$$53_{16} = 5 \times 16 + 3 = 83_{10}$$

$$101,0011_2 = 64 + 16 + 2 + 1 = 83_{10}$$

Problem 23 *Use the new method to convert the number* 100111_2 *to the hexadecimal form.*

Using the conversion table, we get the following.

$$10,0111_2 = 27_{16}$$

Once finished, convert both numbers to the decimal form to check the result.

$$27_{16} = 32 + 7 = 39_{10}$$

$$10,0111_2 = 32 + 4 + 2 + 1 = 39_{10}$$

Problem 24 *Use the new method to convert the number* $1da$ *to the binary form.*

Using the conversion table, obtain the following.

$$1da = 1,1101,1010_2$$

Once finished, convert both numbers to the decimal form to check the result.

$$1da = 256 + 13 \times 16 + 10 = 474_{10}$$

$$1,1101,1010_2 = 256 + 128 + 64 + 16 + 8 + 2 = 474_{10}$$

Question 1 *Eight is also a power of two. Can you guess an efficient way to go back and forth between the binaries and octals?*

Problem 25 *Use the new method to convert the number* 10101_2 *to the octal form.*

Using the conversion table, obtain the following.

$$10,101_2 = 25_8$$

Once finished, convert both numbers to the decimal form to check the result.

$$25_8 = 16 + 5 = 21_{10}$$

$$10,101_2 = 16 + 4 + 1 = 21_{10}$$

Problem 26 *Use the new method to convert the number* 753_8 *to the binary form.*

Using the conversion table, obtain the following.

$$753_8 = 111,101,011_2$$

Once finished, convert both numbers to the decimal form to check the result.

$$753_8 = 7 \times 64 + 40 + 3 = 491_{10}$$

$$111,101,011_2 = 256 + 128 + 64 + 32 + 8 + 2 + 1 = 491_{10}$$