METRIC SPACES

MATH CIRCLE (HS1) 10/20/2013

Definition: Call (M, d) a metric space if M a set and $d : M^2 \to \mathbb{R}$ such that d is a metric.

Recall that we say d is a metric for M if d satisfies the following properties (for all $x, y, z \in M$):

(1) d is non-negative ($d(x, y) \geq 0$),
(2) d is symmetric ($d(x, y) = d(y, x)$),
(3) $x = y$ if and only if $d(x, y) = 0$,
(4) d satisfies the triangle inequality ($d(x, z) \leq d(x, y) + d(y, z)$).

Last time we saw that, in particular, $(\mathbb{R}^2, d_E)$ and $(\mathbb{R}^2, d_T)$ are examples of metric spaces.

For reference, here is the proof that $(\mathbb{R}^2, d_T)$ is a metric space.

Proof. Throughout, we let $P = (x_1, y_1), Q = (x_2, y_2), R = (x_3, y_3)$. Recall $d_T(P, Q) = |x_2 - x_1| + |y_2 - y_1|$. Since absolute values are always positive, Prop. 1 is clear. Furthermore, $|a - b| = |b - a|$ so Prop. 2 is clear as well. For Prop. 3 we have:

$$P = Q \Leftrightarrow x_1 = x_2 \& y_1 = y_2 \Leftrightarrow |x_2 - x_1| = |y_2 - y_1| = 0 \Leftrightarrow |x_2 - x_1| + |y_2 - y_1| = 0 \Leftrightarrow d_T(P, Q) = 0.$$

For Prop. 4 we thus have

$$d_T(P, R) = |x_3 - x_1| + |y_3 - y_1| = |x_3 - x_2 + x_2 - x_1| + |y_3 - y_2 + y_2 - y_1|.$$

Applying the fact $|a + b| \leq |a| + |b|$ twice gives us

$$d_T(P, R) \leq |x_3 - x_2| + |x_2 - x_1| + |y_3 - y_2| + |y_2 - y_1| = d_T(P, Q) + d_T(Q, R)$$

which completes the proof. □

0) Prove that Prop. 1 of being a metric is not really needed. That is, prove that if $d(P, Q)$ satisfies Properties 2,3, and 4, then it automatically satisfies Prop. 1.

1) Let $M = \mathbb{R}$. For each of the following, which properties of a metric does d have? Is (M, d) a metric space?

a) $d(x, y) = |x - y|$

b) $d(x, y) = |x - y| + 1$

c) $d(x, y) = |x^2 - y^2|$

d) $d(x, y) = |x - y|^2$

This handout is based on problems from previous math circle handouts as well as E. Krause "Taxicab Geometry."

e) $d(x, y) = |x - y|^{1/2}$

f) $d(x, y) = |x - y| + |x^2 - y^2|$ Hint: Use the previous parts!

2) A metric on $\mathbb{R}$ is called translation invariant if $d(x, y) = d(x + k, y + k)$ for all $k \in \mathbb{R}$.

a) Give an example of a metric space $(\mathbb{R}, d)$ which is translation invariant.

b) Give an example of a metric space $(\mathbb{R}, d)$ which is not translation invariant.

3) Come up with a definition of a metric d that makes (S, d) into a metric space for ANY S. Hint: Think of d as answering the question "Have I moved?"

An ultrametric is a function that satisfies Properties 1, 2, and 3 of being a metric, as well as property 4': $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. We call (M, d) such that d is an ultrametric an ultrametric space.

4) Show that any ultrametric space is a metric space

5) The metric you defined in 3) is often called the discrete metric. Show that a discrete metric is an ultrametric.

For Further Investigation (Homework):

1) Try to find d so that i) d is not a discrete metric, ii) $(\mathbb{R}, d)$ is an ultrametric space. Hint: d can still look similar to a discrete metric and it is still possible that $d(0, x) = |x|$.

2) Let S be the set containing all binary sequences of length n. (That is, each member of S has the form $x_1 x_2 \cdots x_n$ with each $x_i = 0$ or $x_i = 1$.) Try to come up with a d so that (S, d) is a metric space.