WHAT IS A DISTANCE?

MATH CIRCLE (HS1) 10/13/2013

Recall the Euclidean (d_E) and Taxicab (d_T) distances introduced last time:

$$d_E(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},\ d_T(P,Q) = |x_2 - x_1| + |y_2 - y_1|.$$

Note: Unless noted otherwise, all answers must be proven.

0) Suppose we have a formula $d(P,Q)$ that is suppose to measure some type of distance between P and Q. Below are four basic properties of a distance; for each of them, translate the property into a mathematical statement about d.

a) Distances are never negative.

b) If I travel 0 distance then I haven't moved. Conversly, I don't have to move to travel to the point I'm currently at.

c) The distance from here to there is the same as the distance from there to here.

d) Suppose I want to get from here to there. It is never (strictly) faster to make a stop along the way.

We call a d that satisfies the above four properties a *metric*. As you can probably guess, d_E is in fact a metric (see homework).

1) a) Prove that for any two real numbers x, y, $|x+y| \leq |x| + |y|$. Hint: Consider cases!

b) Prove that d_T is a metric.

Define two more "distance candidates" for the plane $\mathbb{R}^2$:

$$d_L(P,Q) = \max\{|x_2 - x_1|, |y_2 - y_1|\},\ d_S(P,Q) = \min\{|x_2 - x_1|, |y_2 - y_1|\}.$$

2) What do circles look like using d_L and d_S? In particular, graph the circle with center $(0,0)$ and radius 3 using both distances.

3) a) Do you think d_L is a metric? what about d_S?

b) Prove (or disprove) your guesses.

4) Prove that property a) of being a metric is not really needed. That is, prove that if $d(P,Q)$ satisfies properties b), c), and d), then it automatically satisfies a).

5) Recall that the geometric transformation *translation*. A translation of "a units to the right and b units up",can be thought of algebraically as a map $\varphi_{a,b}$ such that $\varphi_{a,b}(P) = (x+a, y+b)$ if $P = (x,y)$.

Prove that for $d = d_E$ and $d = d_T$, then d is *translation invariant,* that is, show that $d(P,Q) = d(\varphi_{a,b}(P), \varphi_{a,b}(Q))$.

This handout is based on problems from previous math circle handouts as well as E. Krause "Taxicab Geometry."

6) So far we have only seen ways that d_E and d_T are similar (they are both metrics and they are both translation invariant). Come up with a property that distinguishes them! Hint: Think of other geometric transformations.

For Further Investigation (Homework):

1) Prove that d_E is a metric. Note, property d) is hard, start by proving the following statements below.

i) Suppose the quadratic $Ax^2 + Bx + C = 0$ has at most one real rool. Prove that $B^2 - 4AC \leq 0$.

ii) Prove (for real numbers a_1, a_2, b_1, b_2)

$$(a_1x + b_1)^2 + (a_2x + b_2)^2 = (a_1^2 + a_2^2)x^2 + 2(a_1b_1 + a_2b_2)x + (b_1^2 + b_2^2) = 0$$

has at most one real root.

iii) Prove a special case of the Cauchy Schwarz Inequality:

$$(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2).$$

iv) Now you are ready to prove property d)!

2) Challenge: Prove the general Cauchy Schwarz Inequality

$$(a_1b_1 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2).$$

Hint: This is not really any harder than what you did above!