LAMC Beginners' Circle October 6, 2013

Oleg Gleizer

oleg1140@gmail.com

Solution made up by Zhihao Zhao

A problem solving session

Problem 1 Put the right sign, >, <, or =, between the fractions.

$$\frac{2012}{2013} \ \square \ \frac{2013}{2014}$$

Solution: Since $\frac{2012}{2013} = 1 - \frac{1}{2013}$, $\frac{2013}{2014} = 1 - \frac{1}{2014}$, and $\frac{1}{2013} > \frac{1}{2014}$, we get $\frac{2012}{2013} < \frac{2013}{2014}$.

Problem 2 The number 2^{1000} is written down in the decimal system. Would the number of its digits exceed 400? Why or why not?

Solution: No. Since $2^{1000} = 2 \times 2^{999} = 2 \times (2^3)^{333} = 2 \times 8^{333} < 2 \times 10^{333} < 10^{399}$, it can be concluded that the number of digits of 2^{1000} will not exceed 400.

Problem 3 The number 2^{1000} is written down in the binary system. What is the number of its digits? How many ones are there among them?

Solution: As ${2^{1000}}_{10} = 100 \ldots 0_2$ with exactly a thousand "0's," as

$${2^{1000}}_{10} = 0 \times 2^0 + 0 \times 2^1 + \cdots + 0 \times 2^{999} + 1 \times 2^{1000}$$

it can be concluded that the number of its digit is 1001, and there is only one "1" in this expression.

Problem 4 A ruler has three marks, zero, seven, and eleven inches.

(1) Is it possible to use the ruler for drawing a segment that is 8" long? (2) How about a 5"-long segment?

Solution: (1) Yes. $8 = (11 - 7) + (11 - 7)$.

First, get 4” by 11”-7”
Second, get 8’’ by 4”+4”
(2) Yes. $5 = (11 - 7) + (((11 - 7) + (11 - 7)) - 7)$.
First, find length 1” by 8”-7”. The 8’’ segment can be obtained from the previous problem.
Second, get 4” by 11”-7”
Third, get 5” by 4”+1”.

Problem 5 Three houses A, B, and C, are built along a straight road.

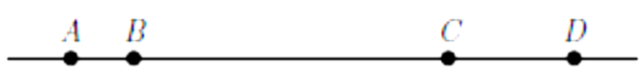
You are an engineer commissioned to find a place for a water well W so that the total distance from W to A, B, and C is the shortest possible. Where would you place the well?

Solution: Point B should be the place to construct the well. Considering the total distance to A and C first: as long as the well is between A and C, the total distance to A and to C is AC. And this should be the minimum total distances for AW+CW.

(Because suppose the well W is between A and C, and AW=x, then CW=AC-x. As a result, AW+CW= x+ AC-x=AC. This is the minimum case because if W is either at A’s left or C’s right, supposing it locates on A’s left and AW=y, then AW+CW=y+AC+y=AC+2$y\geqslant$AC.)

Now start to consider A, B, and C together. AW+BW+CW=AC (fixed)+BW for W between A and C. So the minimum total distance would happen if BW is minimum, i.e. W=B. At this time, AW+BW+CW=AC.

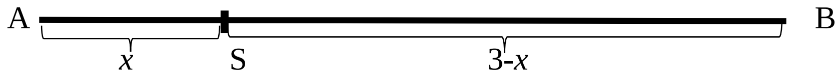
Problem 6 This time they have four houses, A, B, C, and D, located along a straight road.
A B C D
Where would you build the well?



Solution: As we discovered in Problem 5, for points A and D, the total minimum distances occur when W is between A and D. For the same reason, for points B and C, the total minimum distances occur when W is between B and C. So the well should be placed between B and C.

Problem 7 There are 100 schoolchildren living in the town of A. There are 50 schoolchildren living in the town of B. The towns are connected by a straight highway that is 3 miles long. The towns' councils decide to build one school for both towns. Where should they locate it so that the total distance of travel for all the 150 students summed together is minimal?

Solution: Suppose the school S locates in between A and B with AS=*x*,



The total distances for the 100 students living in A town is 100·*x*. The total distances for the 50 students living in B town is 50·(3-*x*). The total distances are 100*x*+50(3-*x*) = 150+50*x*.

So, when *x*=0, i.e. the school is built at town A, the total distances for the 150 students equals 150, achieving the minimum value.

The following problem are discussing cases of "catching-up." In order to help you understand it, please think over this example first.

Example: Suppose there are two people A and B starting to travel in the same direction (A is behind B) from two places with distance of 100 miles in between. The speed of A is 3 miles/hour, and the speed of B is 2 miles/hour.

When do A catch up with B?

Solution: Suppose at time t hrs, A catches up with B.
Then the distance A has traveled is 3m/hr×t hrs, and B has traveled is 2m/hr×t hrs.

Since they are moving at the same direction, when A catches up with B, the difference in distances traveled is 100 miles. So,

$$3\mathrm{t} - 2\mathrm{t} = 100$$
$$\mathrm{t} = 100 \text{ hrs}$$

Note that this is equivalent of saying that A catches up with B in 1 mile per hour, and there are 100 miles for A to catch up. So, we can directly compute

$$\mathrm{t} = \frac{initial\ distance}{difference\ in\ speed} = \frac{100}{3-2} = 100.$$

Problem 8 At noon, the hour and minute hands of a clock point in the same direction. When would they point in the same direction next time?

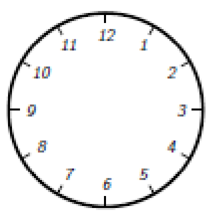
Solution: This is a variant of the example above. We can think the minute had is lagged 1 round behind the hour hand. Also note that the speed of minute hand is 1 round/hour, while the speed of the hour hand is $\frac{1}{12}$ round/hour. So the catching-up time is

$$t = \frac{initial\ distance}{difference\ in\ speed} = \frac{1}{1 - \frac{1}{12}} = \frac{12}{11}.$$

Problem 9 How many times per day (24 hrs.) do the hour and minute hand point in the same direction?

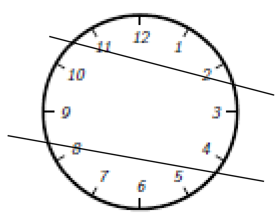
Solution: Use the result we get from last problem, we know that in 24 hours, the minute and hour hands can meet $\frac{24}{\frac{12}{11}} = 22\ times$. Also, don't forget to include the initial situation at 12 am, when the minute and hour hands point in the same direction. So the final answer should be 22+1=23 times.

Problem 10 Divide a clock face with two straight lines so that the sums of the numbers in each part are equal.

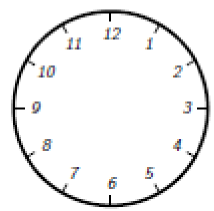


Solution: First, consider the sum of 1~12 is $\frac{(1+12)\times 12}{2} = 78$. Since two lines can divide a plane in three or four parts, and 78 is divisible by 3 not by 4, we can conclude that the clock shoul be divided into three parts, each part having a sum of 26.

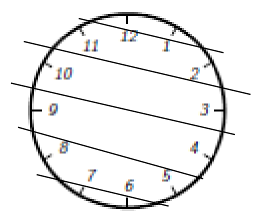
One possible outcome is:



Problem 11 Divide a clock face into six parts so that each part contains two numbers and the six sums of two numbers are equal.



Solution: Similar to the previous problem, each of the six sums would be $\frac{78}{6} = 13$. One possible division would be:



Problem 12 Find the following sum.

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{99 \times 100} =$$

Solution: Use the technique of telescoping:

$$\frac{1}{1 \times 2} = 1 - \frac{1}{2}$$
$$\frac{1}{2 \times 3} = \frac{1}{2} - \frac{1}{3}$$
$$\frac{1}{3 \times 4} = \frac{1}{3} - \frac{1}{4}$$
$$\cdots\cdots$$
$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Quick proof: $\frac{1}{n} - \frac{1}{n+1} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)}$.

So,

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{99\times 100}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{99}-\frac{1}{100}\right)$$

Note that the intermediate term $\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{99}$ will be cancelled out, so

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{99\times 100}=1-\frac{1}{100}=\frac{99}{100}$$

> *Definition:*
> Let n be an integer and M be a positive integer. We denote by
> $$r \equiv n \pmod{M}$$
> the remainder *r* when n is divided by M, that is,
> $$n = q \times M + r$$
> where *q* is an integer and $0 \le r < M$.
>
> Also, please note that $m \equiv n \pmod{M}$ is equivalent to the notion that *m* and *n* have the same remainder if divided by M.

Problem 13 Prove that an integral number is divisible by three if and only if the sum of its digits is divisible by three.

Solution: Suppose we have a finite digit number $\overline{a_n a_{n-1}\cdots a_1 a_0}$,

$$\begin{aligned}\overline{a_n a_{n-1}\cdots a_1 a_0} &= 1\times a_0 + 10\times a_1 + \cdots 10^n \times a_n\\ &= 1\times a_0 + (1+3\times 3)\times a_1 + \cdots + (1+3\times 33\cdots 3)\times a_n\\ &= 1\times a_0 + 1\times a_1 + 3\times 3\times a_1 + \cdots + 1\times a_n + 3\times 33\cdots 3\times a_n\\ &= (a_0+a_1+\cdots+a_n) + 3\times(3\times a_1 + 33\times a_2 + \cdots + 33\cdots 3\times a_n)\end{aligned}$$

By definition of module, since $[3\times(3\times a_1 + 33\times a_2 + \cdots + 33\cdots 3\times a_n)]$ is divisible by 3, we get that

$$\overline{a_n a_{n-1}\cdots a_1 a_0} \equiv a_n + a_{n-1} + \cdots + a_1 + a_0 \pmod{3}.$$

Problem 14 Prove that an integral number is divisible by nine if and only if the sum of its digits is divisible by nine.

Solution: Similarly, suppose we have a finite digit number $\overline{a_n a_{n-1}\cdots a_1 a_0}$,

$$\begin{aligned}\overline{a_n a_{n-1}\cdots a_1 a_0} &= 1\times a_0 + 10\times a_1 + \cdots 10^n \times a_n\\ &= 1\times a_0 + (1+9\times 1)\times a_1 + \cdots + (1+9\times 11\cdots 1)\times a_n\\ &= 1\times a_0 + 1\times a_1 + 9\times 1\times a_1 + \cdots + 1\times a_n + 9\times 11\cdots 1\times a_n\\ &= (a_0+a_1+\cdots+a_n) + 9\times(1\times a_1 + 11\times a_2 + \cdots + 11\cdots 1\times a_n)\end{aligned}$$

By definition of module, since $[9\times(1\times a_1 + 11\times a_2 + \cdots + 11\cdots 1\times a_n)]$ is divisible by 9, we get that

$$\overline{a_n a_{n-1} \cdots a_1 a_0} \equiv a_n + a_{n-1} + \cdots + a_1 + a_0 \text{ (mod 9)}.$$

Problem 15 Oleg multiplied

$$1 \times 2 \times 3 \times \cdots \times 100$$

and added up all the digits of the resulting number. Then he added up all the digits of the sum and then he proceeded doing so until he got down to a one-digit number. What was the one-digit number?

Solution: Define "SUM" to be an operation that adds up all the digits of the resulting number. Consider the conclusion we got from Problem 14, for any number x, we get that

$$x \equiv \text{SUM } x \text{ (mod 9)}$$

Thus,

$$1 \times 2 \times 3 \times \cdots \times 100 = 100! \equiv \text{SUM } 100! \text{ (mod 9)} \equiv \text{SUM(SUM } 100!)\text{(mod 9)} \equiv \cdots$$
$$\equiv \text{SUM}(\cdots(\text{SUM } 100!)\cdots) \text{ (mod 9)}$$

Note that the one-digit number is obtained exactly by adding up all the digits of the sum as many times as possible, so $\text{SUM}(\cdots(\text{SUM } 100!)\cdots)$ is what we are looking for, and

$$100! \equiv \text{SUM}(\cdots(\text{SUM } 100!)\cdots) \text{ (mod 9)}$$

To compute the remainder of 100! Divided by 9, we also note that 100! is divisible by 9, because

$$100! = 1 \times \cdots \times 9 \times \cdots \times 100.$$

By definition of module, we have $1 \times 2 \times 3 \times \cdots \times 100 \equiv 0 \text{ (mod 9)}$.
So, the one-digit number can only be 0 or 9. Since it's a sum of digits (which are all non-negative and cannot be all zero), 0 is not possible. Thus the answer is 9.

Problem 16 The decimal representation of a number has one hundred digits zero, one hundred digits one, and one hundred digits two. Can the number be a perfect square? Why or why not?

Solution: Consider all the possible remainders of squares divided by 9:

$$0^2 \equiv 0 \text{ (mod 9)}$$
$$1^2 \equiv 1 \text{ (mod 9)}$$
$$2^2 \equiv 4 \text{ (mod 9)}$$
$$3^2 \equiv 0 \text{ (mod 9)}$$
$$4^2 \equiv 7 \text{ (mod 9)}$$
$$5^2 \equiv 7 \text{ (mod 9)}$$
$$6^2 \equiv 0 \text{ (mod 9)}$$
$$7^2 \equiv 2 \text{ (mod 9)}$$
$$8^2 \equiv 1 \text{ (mod 9)}$$

i.e. all the squares should have remainder among 0,1,2,4,7.
This list is exhaustive because

$$x^2 \equiv r^2, \quad \text{if and only if } x \equiv r \text{ (mod 9)}$$

However, the given number $\equiv 1 \times 100 + 2 \times 100 \equiv 300 \equiv 3 \text{ (mod 9)}$, according to Problem 14. Because 3 is not on the list, the given number cannot be a square.

Problem 17 Find the value of 2×3 in the place-value system where 2 + 3 = 11.

Solution: Since $2_{10} + 3_{10} = 5_{10} = 11_b$, we know that $5_{10} = 1 \times b^0 + 1 \times b^1 = 1 + b$.

Thus, $b = 4$.

$$2 \times 3 = 6_{10} = 2 \times 4^0 + 1 \times 4^1 = 12_4.$$

The following problem was communicated to me by one of our students, Ethan Kogan.

Problem 18 Make a hundred using only four nines.

Solution: $100 = 99 + \frac{9}{9}$

Problem 19 Prove the following polynomial identity.

$$x^2 + y^2 = (x + y)\ (x - y)$$

Solution:

$$\text{RHS} = (x + y)\,(x - y) = x^2 + xy - xy + y^2 = x^2 + y^2 = \text{LHS}$$

Recall that a positive integer is called prime if it has only two positive integral factors, itself and one. For example, the number five is prime. The number six is not prime, since $6 = 3 \times 2$. A positive integer that is not prime is called composite.

Problem 20 Prove that the number 999,991 is not prime.

Solution: $999{,}991 = 1{,}000{,}000 - 9 = 1000^2 - 3^2 = (1000 - 3)(1000 + 3) = 997 \times 1003$. So 999,991 is not a prime.

Problem 21 Prove the following polynomial identity.

$$x^3 + y^3 = (x + y)\ (x^2 - xy + y^2)$$

Solution: Start from the right-hand side (RHS)

$$\text{RHS} = (x + y)\,(x^2 - xy + y^2) = x^3 + x^2y - x^2y - xy^2 + xy^2 + y^3 = x^3 + y^3 = \text{LHS}$$

Problem 22 Prove that the number 1001_b is composite for any base b.

Solution:

$$1001_b = (1 \times b^0 + 0 \times b^1 + 0 \times b^2 + 1 \times b^3)_{10} = (b^3 + 1)_{10} = (b + 1)(b^2 - b + 1)_{10}$$

Since b+1, is greater than 1, 1001_b is a composite no matter what value b is.

If you are finished solving all the above problems, please fill out the hexadecimal multiplication *table below.*

Solution:

x	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10
1	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10
2	2	4	6	8	A	C	E	10	12	14	16	18	1A	1C	1E	20
3	3	6	9	C	F	12	15	18	1B	1E	21	24	27	2A	2D	30
4	4	8	C	10	14	18	1C	20	24	28	2C	30	34	38	3C	40
5	5	A	F	14	19	1E	23	28	2D	32	37	3C	41	46	4B	50
6	6	C	12	18	1E	24	2A	30	36	3C	42	48	4E	54	5A	60
7	7	E	15	1C	23	2A	31	38	3F	46	4D	54	5B	62	69	70
8	8	10	18	20	28	30	38	40	48	50	58	60	68	70	78	80
9	9	12	1B	24	2D	36	3F	48	51	5A	63	6C	75	7E	87	90
A	A	14	1E	28	32	3C	46	50	5A	64	6E	78	82	8C	96	A0
B	B	16	21	2C	37	42	4D	58	63	6E	79	84	8F	9A	A5	B0
C	C	18	24	30	3C	48	54	60	6C	78	84	90	9C	A8	B4	C0
D	D	1A	27	34	41	4E	5B	68	75	82	8F	9C	A9	B6	C3	D0
E	E	1C	2A	38	46	54	62	70	7E	8C	9A	A8	B6	C4	D2	E0
F	F	1E	2D	3C	4B	5A	69	78	87	96	A5	B4	C3	D2	E1	F0
10	10	20	30	40	50	60	70	80	90	A0	B0	C0	D0	E0	F0	100