How to Win in *Nim*

UCLA Math Circle
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A combinatorial game is a game that satisfies the following conditions:

1. There are two players.
2. There is a set, usually finite, of possible positions of the game.
3. The rules of the game specify for each player and position which moves are legal. If the rules makes no distinction between players, the game is called **impartial**. Otherwise, it is call **partisan**.
4. The players alternate to make moves.
5. The game ends when a position with no possible further moves is reached. Under **normal play rule**, the last player to move wins. Under **misère play rule**, the last player to move loses.
6. The game ends in a finite number of moves no matter how it is played. This is called the **Ending Condition**.

A position is called a P**-position** if the previous player (the person just played) has a winning strategy. A position is called a N**-position** if the next player (the next person to play) has a winning strategy. Here are some properties of P-position and N-position

1. All terminal positions are P-positions.
2. From every N-position, there is at least one move to a P-position.
3. From every P-position, every move is to an N-position.

Here are two examples of combinatorial games:

Game 1: Subtraction There are 24 chips in a pile. Two players take turns to remove chips from the pile. Each player can either remove 1, 2 or 3 chips from the pile in one move. The player to remove the last chip wins.

Game 2: Nim There are three piles of chips, represented by (a_1, a_2, a_3). Two players take turns to remove chips from the pile. During each turn, a player choose a pile, and can remove as many chips (at least one chip) as they wish from that pile. The player to remove the last chip overall wins.

Definition 1. *Let x, y be positive integers with binary representations $(x_m \dots x_0)_2$ and $(y_m \dots y_0)_2$ respectively. Then the **nim-sum** of x and y is the integer z with the binary representation $(z_m \dots z_0)_2$ where*

$$z_k \equiv x_k + y_k \mod 2.$$

We write $z = x \oplus y$.

Theorem 1. *A position (a_1, a_2, a_3) in Nim is a P-position if and only if the nim-sum of the components is zero, i.e.*

$$a_1 \oplus a_2 \oplus a_3 = 0.$$

Definition 2. *A **directed graph**, G is a pair (X, F) where X is a nonempty set of vertices representing positions and F is a function that gives for each $x \in X$ a subset of X, say $F(x) \subset X$, representing the the positions that a player could move to from position x. If $F(x)$ is empty, then x is called a terminal position.*

Definition 3. *Let S be a set of non-negative integers. Then the **minimal excludant**, or **mex**, of S is the smallest non-negative integer* not *in S.*

Definition 4. *The **Sprague-Grundy function** of a graph, (X, F) is a function g defined on X taking non-negative integer values, such that*

$$\begin{aligned} g(x) &= \min\{n \geq 0 : n \neq g(y) \text{ for } y \in F(x)\} \\ &= \operatorname{mex}\{g(y) : y \in F(x)\}. \end{aligned}$$

Definition 5. *Given two combinatorial games $G_1 = (X_1, F_1), G_2 = (X_2, F_2)$, their sum $G = (X, F)$ is a combinatorial game defined by*

$$\begin{aligned} X &= X_1 \times X_2, \\ F(x_1, x_2) &= (F_1(x_1) \times \{x_2\}) \cup (\{x_1\} \times F_2(x_2)). \end{aligned}$$

Theorem 2. *If g_i is the Sprague-Grundy function of $G_i, i = 1, 2$, then $G = G_1 + G_2$ has Sprague-Grundy function $g(x_1, x_2) = g_1(x_1) \oplus g_2(x_2)$.*