Week 3 solutions!

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Warm up

In each of the following, we let x represent the instructor's favorite number.

- $5x + 1 = 5$ so $5x = 4$, and dividing by 5, $x = \frac{4}{5} = .8$.
- $10x = 6 + x$ so $9x = 6$ and hence $x = \frac{6}{9} = .66666\cdots$.
- $100x = 100 + x$, hence $99x = 100$, so $x = \frac{100}{99} = 1.010101\cdots$.
- $1000x = 37 + x$, so $999x = 37$ or $x = \frac{37}{999} = .037037037\cdots$.

Problems

1. Yes there is something to prove. What we proved before is called the converse of this statement. That is if we let a represent the phrase "x is a fraction with denominator having only powers of 2 and 5" and b represent the phrase "x is a decimal which terminates". We have proven

   $$a \Rightarrow b$$

   what we are looking to prove now is

   $$b \Rightarrow a$$

   also called the converse, which is different.

2. (a) $\frac{89}{100}$
   (b) $\frac{35}{100} = \frac{7}{20}$
   (c) $\frac{125}{10000} = \frac{1}{80}$
   (d) $\frac{4444}{100} = \frac{1111}{25}$

3. Given an arbitrary terminating decimal x we can write

   $$x = .a_1 a_2 \cdots a_k 0 \cdots$$

   then $10^k x = a_1 a_2 \cdots a_k$ an integer.

4. In the previous problem we have shown that $10^k x$ is an integer, says r. So $10^k x = r$, thus $x = \frac{r}{10^k}$ and we have the claim.

5. (a) $.1111\cdots$
   (b) $.2222\cdots$
   (c) $.3333\cdots$
   (d) $.4444\cdots$
   (e) $.5555\cdots$

(f) $.6666\cdots$

(g) $.7777\cdots$

(h) $.8888\cdots$

(i) Well, we saw in (a) that $\frac{1}{9} = .1111\cdots$ so $1 = 9\cdot\frac{1}{9} = .9999\cdots$. Thus we have two decimal representations for the number 1.

6. (a) $3.333\cdots = 3 + .3333\cdots = 3 + \frac{1}{3} = \frac{10}{3}$

   (b) $17.5555\cdots = 17 + .5555\cdots = 17 + \frac{5}{9} = \frac{17*9+5}{9}$

   (c) $2.08888\cdots = 2 + \frac{1}{10}\cdot .8888\cdots = 2 + \frac{1}{10}\cdot\frac{8}{9} = \frac{188}{90}$

7. (a) $10x = 2.222\cdots$

   (b) $10x = x + 2$

   (c) $9x = 2$, so $x = \frac{2}{9}$ (which we already knew from the previous problem..)

   (d) this is just telling you to realize that in part (c) what we have shown is that $.222\cdots = \frac{2}{9}$.

8. $10x = 6 + x$, so we get $9x = 6$, or $x = \frac{6}{9}$.

9. (a) $10w = 4.666\cdots$ and $10w - 4 = .6666\cdots$

   (b) Well, we just showed $.666\cdots = \frac{2}{3}$ so we have that as the fractional version of $10w - 4$.

   (c) Therefore $10w = 4 + \frac{2}{3} = \frac{14}{3}$ and so $w = \frac{14}{30}$.

10. (a) $100y = 36.363636\cdots$

    (b) $100y = 36 + y$

    (c) Therefore, we get $99y = 36$ or $y = \frac{36}{99}$.

    (d) Again, this is just looking at part (c) and realize we have derived $.36363636 = \frac{36}{99}$.

11. $100y = 5.05050505 = 5 + y$. Therefore, we solve this and get $99y = 5$, or $y = \frac{5}{99}$.

12. The idea is to multiply by 10 to the power of (length of period of the decimal). Therefore we should multiply by 10^3 or 1000. We get $1000y = 148.148148\cdots$, and hence $1000y = y + 148$, so $999y = 148$ and $y = \frac{148}{999}$.

13. Let y represent the decimal number. Multiply by 1000, and we get $1000y = abc.abcabc\cdots$, that is

$$1000y = abc + y$$

solving this equation we get $999y = abc$ and hence $y = \frac{abc}{999}$.

14. (a) Yes, $5242362343 = \frac{5242362343}{1}$

    (b) Yes, we saw that every terminating decimal comes from a fraction, so 47.26 comes from a fraction, and so -47.26 also comes from a fraction (just multiply the numerator by -1).

    (c) Yes, but we haven't actually proven this yet and its quite hard. Heres the proof: Let x be an arbitrary eventually periodic decimal, then

$$x = .a_1\cdots a_k b_1\cdots b_n b_1\cdots b_n b_1\cdots b_n\cdots$$

so $a_1,\cdots,a_k$ are random and the pattern starts at $b_1,\cdots,b_n$. Multiplying x by 10^k we get

$$10^k x = a_1\cdots a_k + .b_1\cdots b_n b_1\cdots b_n\cdots$$

Now, preforming the same process as in problem 13, but with n digits instead of three, we see that

$$.b_1\cdots b_n b_1\cdots b_n\cdots = \frac{b_1\cdots b_n}{10^n - 1}$$

So adding it up we get

$$10^k x = a_1 \cdots a_k + \frac{b_1 \cdots b_n}{10^n - 1}$$

and solving this for x we see that x is indeed a fraction.

(d) No, you came up with some examples yourself last week of decimals that did not repeat, and we have already proven that every rational has a decimal equivalent which repeats.

Homework

1. (a) No, it does not repeat.

   (b) Yes! we can bring the 7^2 and 9 down into the big denominator making this into a fraction with integers on the top and bottom.

2. Yes, you can do this! (it sounds a little bit surprising, but the once you see the answer you will say D'oh! like Homer Simpsons)

   Let $a = .010100100\cdots$ (which we know is not rational by the first homework!) and set $b = -a$, then b is also not rational because it still has a decimal which does not repeat. But

   $$a + b = a + (-a) = a - a = 0$$

   and of course, $0 = \frac{0}{1}$ is rational.

3. If we denote the number by q, multiply the number by 1000 we get

   $$1000q = def + .abcabc\cdots$$

   we already know the fractional equivalent of this decimal, from the worksheet so we get

   $$1000q = def + \frac{abc}{999}$$

   hence

   $$q = \frac{999(def) + abc}{999 \cdot 1000}$$