SLAM InDUNCtion!

Math Circle (Advanced)

October 13, 2013

1. Simplify the sum $1 + 2 + 3 + ... + n$. (Try to come up with a way to argue your answer is correct if you can!)

The Method of Mathematical Induction (MMI) is a way to show that a property $P(n)$ holds for all $n = 1, 2, 3, ...$.
A proof using MMI consists of two steps:
BASIS: Show that $P(1)$holds.
INDUCTION: Assume $P(k)$ holds, and show that $P(k+1)$ holds.
Then, MMI tells us that $P(n)$ holds for all n.

2. Using MMI, prove your answer for Problem 1.

3. Prove that $1 + 3 + 4 + ... + (2n + 1) = n^2$...

   (a) using mathematical induction.

   (b) using only the equation proved in Problem 1 (no induction).

4. The steps of mathematical induction (basis, and induction) need not be performed in this order. (Can you explain why?) For this problem, consider the following statement: For all n, $n + 1 < n$.

   (a) This time, let's do the induction step first. Assume that $P(k)$ holds, and prove that $P(k + 1)$ holds.

   (b) Does something seem fishy about your result from part (a)? Is the statement actually true?

(c) Has induction failed? Or can you **explain** why, despite the fact that you were able to prove what you did about $P(k+1)$, that the statement could still be false?
(Hint: Your induction proof is not complete! What's missing?)

5. Is it always true that for $n \geq 1$, $3^n > n^2$? If so, prove it. If not, list a counterexample.

6. Is it always true that for $n > 3$, $3^n > 3n^2$? If so, prove it. If not, list a counterexample.

7. $2n$ dots are placed around the outside of the circle. n of them are colored red and the remaining n are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the correct point.

8. Is $8^n - 1$ always divisible by 7? If so, prove it. If not, list a counterexample.

9. Use MMI to prove that it is possible to cover a 2^n x 2^n grid with L-shaped triominoes, provided the top right corner square is omitted from each grid.

10. Prove or disprove: $n^2 + n + 41$ is prime for all n.

11. Using MMI, prove that $(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n \cdot (n+1) = \frac{1}{3}(n)(n+1)(n+2)$.

12. Consider a collection of intersecting lines on the plane. Assume that the lines divide the plane into as many regions as possible.

   (a) How many regions of the plane are formed by

      i. 3 such lines?

      ii. 5 such lines?

      iii. 8 such lines?

   (b) In the first two cases, Jeff found a way to color the entire plane with only two different colored pens, with no two adjacent regions having the same color. (Regions that share a side are considered adjacent; regions that meet at a point are not.) Verify Jeff's work for the case of

      i. 3 lines

      ii. 5 lines

(c) While Jeff was busy drawing a meticulous picture to find out if he could color a figure with 8 lines under the same limitations, Angela exclaimed, “Whoa, wait. I already know that you can do it no matter how many lines you’re working with!”
Can you prove Angela correct?