"TAXICAB" GEOMETRY 2

MATH CIRCLE (HS1) 10/06/2013

Recall the Euclidean (d_E) and Taxicab (d_T) distances introduced last time:

$$d_E(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},\ d_T(P,Q) = |x_2 - x_1| + |y_2 - y_1|.$$

Note: Unless noted otherwise, all answers must be proven.

1) Let $\mathcal{C}$ be a (regular) circle with center $(0,0)$ and radius 1. Give examples of circles that intersect it: nowhere, once, twice, and infinitely often.

2) Using Problem 1 as a starting place, examine how taxicab circles can intersect. How does this compare/contrast to circles?

3) Suppose you are given points P, Q. What does the set of points $\{R|d(P,R)+d(R,Q) = d(P,Q)\}$ look like for both $d = d_E$ and $d = d_T$? Start by looking at a few explicit examples.

4) For each set of points P, Q, graph $\{R|d(P,R) = d(Q,R)\}$ for both $d = d_E$ and $d = d_T$.

a) $P = (-2,0), Q = (2,0)$.

b) $P = (0,0), Q = (4,2)$.

c) $P = (-1,-1), Q = (2,2)$.

d) Describe (in your own words) how to do this for arbitrary P, Q. Challenge: Prove your result analytically when $d = d_E$.

5) Let $P = (-3,0)$ and $Q = (3,3)$.

a) Graph $\{R|d_E(P,R) = 2 \cdot d_E(Q,R)\}$. What geometric shape do you get? Challenge: Prove this analytically!

b) Graph $\{R|d_T(P,R) = 2 \cdot d_T(Q,R)\}$. Compare this to the answer in a).

6) a) Suppose that P, Q are points, and r is a positive real number. What shape is

$$\{R|d_E(P,R) + d_E(Q,R) = r\}?$$

b) Challenge: Prove a) analytically if $P = (0,0), Q = (1,0)$ and $r = 4$.

7) Let $P = (-2,1), Q = (2,2)$. Graph the following, for both $d = d_E$ and $d = d_T$:

a) $\{R|d(P,R) + d(Q,R) = 9\}$.

b) $\{R|d(P,R) + d(Q,R) = 13\}$.

This handout is based on problems from previous math circle handouts as well as E. Krause "Taxicab Geometry."

Suppose P is a point, and ℓ is a line. By $d(P, \ell)$ we denote the shortest distance from P to ℓ i.e.

$$d(P, \ell) = \min_{Q \in \ell} \{d(P, Q)\}.$$

where $d = d_E$ or $d = d_T$.

8) Suppose $P = (-1, 1)$. For each of the following lines ℓ, find $d(P, \ell)$ and find all points $Q \in \ell$ such that $d(P, Q) = d(P, \ell)$. (As usual, use both $d = d_E$ and $d = d_T$.)

a) ℓ is the line between $(3, 4)$ and $(2, 2)$.

b) ℓ is the line through $(0, 2)$ with slope $\frac{1}{3}$.

c) ℓ is the line $y = x$.

9) Describe, both geometrically and analyically, the general method to solve the previous problem for arbitrary P and ℓ.

For Further Investigation (Homework):

During the last two weeks we have reviewed the geometric definitions of circle ($\{Q|d(P, Q) = r\}$ for fixed P, r) and ellipse ($\{R|d(P, R) + d(Q, R) = r\}$ for fixed P, Q, r), as well as examined what these shapes look like in taxicab geometry.

Recall that circles and ellipses are two examples of *conic sections*. The other two types of conic sections are *hyperbolas* and *parabolas*. Write down the geometric definitions of these two conic sections, and compare/contrast hyperbolas and parabolas in regular geometry and taxicab geometry.