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Warm-up

Problem 1 *Simplify the following expressions as much as possible.*

$$a. \quad \frac{9^3}{3^6} = \frac{9^3}{(3^2)^3} = \frac{9^3}{9^3} = 1$$

$$b. \quad \frac{(2^3)^4}{2^3 \times 2^4} = \frac{2^{12}}{2^7} = 2^5 = 32$$

$$c. \quad -\frac{2^{3 \times 4}}{2^3 - 2^4} = -\frac{2^{12}}{2^3(1 - 2)} = 2^9 =$$

512

$$d. \quad \frac{2^{3^4}}{2^3 \times 2^4} = \frac{2^{81}}{2^7} = 2^{74}$$

The following piece of the Warm-up section is a part of the book “Algebra Can Be Fun” by Yakov Perelman.

Problem 2 *Write the largest number you can using three digits nine and no other symbols.*

$$9^{9^9}$$

Problem 3 *Write the largest number you can using three digits two and no other symbols.*

$$2^{2^2}$$

Problem 4 *Write the largest number you can using three digits eight and no other symbols.*

$$8^{8^8}$$

Problem 5 Write the largest number you can using three digits three and no other symbols.

$$3^{33}$$

Problem 6 For each of the remaining digits (4,5,6,7) write the largest number you can construct using one digit three times and not using any other symbols.

$$4^{4^4}, 5^{5^5}, 6^{6^6}, 7^{7^7}$$

Place-Value Numerals

The numeral system we use in everyday life is *place-value base ten*. This means that we use ten digits,

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

and that the value of a digit depends on its position in a number. For example the digit 5 means 5 tens in the first and 5 million in the second number below.

$$254 \qquad 75,642,009$$

In this mini-course, we are going to study the place-value systems with the bases two (binaries), three (trinarities), eight (octal numbers), and sixteen (hexadecimal or just hex). To distinguish between the systems, we will write the base as a subscript following the number. For example, the number 254 considered in the standard, decimal, system is written as 254_{10} . This reads as *two hundred fifty four base ten*.

In a place-value system, the numbers are represented as sums of powers of the base. For example,

$$254_{10} = 2 \times 10^2 + 5 \times 10^1 + 4 \times 10^0.$$

Similarly,

$$254_8 = 2 \times 8^2 + 5 \times 8^1 + 4 \times 8^0$$

and

$$254_{16} = 2 \times 16^2 + 5 \times 16^1 + 4 \times 16^0.$$

Problem 7 Find the decimal values of 254_8 and 254_{16} .

$$\begin{aligned} 254_8 &= 2(8)^2 + 5(8) + 4 \\ &= 128 + 40 + 4 = 172_{10} \end{aligned}$$

$$\begin{aligned} 254_{16} &= 2(16)^2 + 5(16) + 4 \\ &= 512 + 80 + 4 = 596_{10} \end{aligned}$$

Question 1 Does the symbol 254 make sense in the trinary place-value system? Why or why not?

If we let the symbols “5” and “4” have the same value which we give them in the decimal place-value system, then **NO**. This is because the values 4_{10} and 5_{10} are written as 11_3 and 12_3 , respectively, in the trinary place-value system; the only digits which are used in this system are 0, 1 and 2. (However, if we were to reassign the symbols “4” and “5” to 0 and 1, then the symbol 254 could in theory make sense.)

Problem 8 *Fill out the table below.*

n	3^n	2×3^n
0	1	2
1	3	6
2	9	18
3	27	54
4	81	162
5	243	486

Then find the value of 254_{10} in the trinary system.

$$254_{10} = 3^5 + 3^2 + 2 = 100102_3$$

As we can see, the number of digits that make sense in a place-value numeral equals to the base. In the binary system, we only have two digits, 0 and 1.

Problem 9 *Fill out the following table.*

n	2^n
0	1
1	2
2	4
3	8
4	16

n	2^n
5	32
6	64
7	128
8	256

Then find the binary representation of 254_{10} .

$$254_{10} = 11111110_2$$

Problem 10 Find the binary representations of the following numbers. Use the table from Problem 9 if needed.

a. $1_{10} = 1_2$

b. $2_{10} = 10_2$

c. $3_{10} = 11_2$

d. $4_{10} = 100_2$

e. $8_{10} = 1000_2$

f. $10_{10} = 1010_2$

g. $65_{10} = 1000001_2$

h. $100_{10} = 1100100_2$

Binary numbers and Egyptian multiplication

Kingdoms of ancient Egypt span the part of human history that begins around 3100 BC and ends at 30 BC when Egypt was conquered by the Roman Empire. The Egyptians were the teachers of the Greeks who in their turn laid foundation for the Western civilization in general and mathematics in particular. That's how the teachers of our teachers have written numbers.

1	10	100	1,000	10,000	100,000	1,000,000 or many
	∩	∩	∩	∩	∩ or ∩	∩

For example, they would have written 2013 as

$$\begin{array}{cc} \cap & \cap \\ \cap & ||| \end{array}$$

The most efficient way to multiply big numbers known today, long multiplication, is based on the fact that our numeral system is place-value. The Egyptian system obviously wasn't. Still, the Egyptians were able to multiply integers quite efficiently in the following manner.

Example 1 *To multiply 57 and 235 the Egyptian way, let us write 1 under the smaller factor, 57. Let us rewrite the larger factor, 235, underneath itself. Let us find the doubles of both numbers and write them down below the original numbers. Let us keep doubling until the number in the left-hand side column*

exceeds 57. The process will result in the following table.

57	235
1	235
2	470
4	940
8	1880
16	3760
32	7520
64	

64 is greater than 57, so we are not going to use this number any further. It only serves us as a stop sign.

The largest number in the left-hand side column less than or equal to 57 is 32. Let us subtract that latter from the former.

$$57 - 32 = 25$$

The largest number in the left-hand side column less than or equal to 25 is 16. Let us subtract again.

$$25 - 16 = 9$$

The largest number in the left-hand side column less than or equal to 9 is 8.

$$9 - 8 = 1$$

Finally, the largest number in the left-hand side column less than or equal to 1 is 1.

$$1 - 1 = 0$$

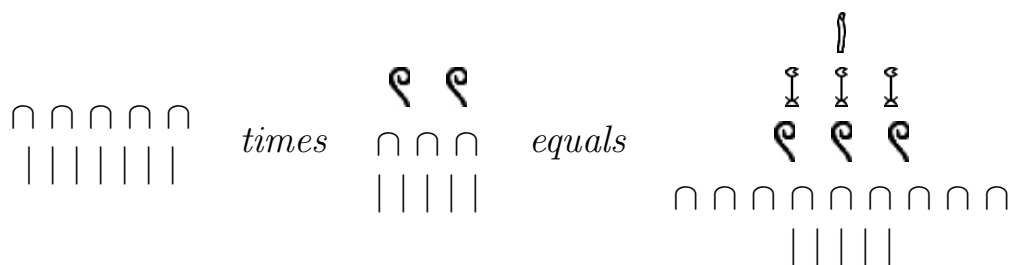
Now we can represent the smaller factor, 57, as a sum of the numbers from the left-hand side column.

$$57 = 32 + 16 + 8 + 1$$

Adding up the corresponding numbers from the right-hand side column renders the desired product.

$$7520 + 3760 + 1880 + 235 = 13,395 = 57 \times 235$$

Or, in Egyptian notations,



Problem 11 To check the answer, multiply 57 and 235 using long multiplication.

$$\begin{array}{r}
 235 \\
 \times 57 \\
 \hline
 1645 \\
 1175 \\
 \hline
 13395
 \end{array}$$

Question 2 *Is it possible that we just got lucky with the numbers from Example 1? Does the trick always work?*

Let's check.

Problem 12 *Use Egyptian multiplication to find the product of 36 and 154. Then use long multiplication to check your answer.*

36	154	
1	154	
2	308	× 154
4	616	× 36
8	1232	-----
16	2464	924
32	4928	462

		5444

$$4928 + 616 = 5544$$

The trick seems to be working!

Question 3 *Why does it work?*

Solving the following problem will help us answer the question.

Problem 13 *Find the binary representations of the following decimal numbers.*

a. $57_{10} = 111001_2$

b. $36_{10} = 100100_2$

Compare your computations to those in Example 1 and Problem 12. Then try to answer Question 3.

Problem 14 Find the binary representations of the following numbers.

$$\begin{aligned} a. \quad 57_8 &= 5(8) + 7 = 47_{10} \\ &= 32 + 0 + 8 + 4 + 2 + 1 = 101111_2 \end{aligned}$$

$$\begin{aligned} b. \quad 36_8 &= 3(8) + 6 = 30_{10} \\ &= 16 + 8 + 4 + 2 + 0 = 11110_2 \end{aligned}$$

Problem 15 Find a trinary representation of the following binary number.

$$\begin{aligned} 1101_2 &= 8 + 4 + 0 + 1 = 13_{10} \\ &= 9 + 3 + 1 = 111_3 \end{aligned}$$

There are enough decimal digits for the binary, trinary, and octal numbers. We shall need some extra digits for the hexs (base 16).

Conversion table

binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111	10000
trinary	0	1	2	10	11	12	20	21	22	100	101	102	110	111	112	120	121
octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17	20
decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
hex	0	1	2	3	4	5	6	7	8	9	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	10

Problem 16 Convert the following hexadecimals to the decimal form.

$$a. \quad 5f_{16} = 5(16) + 15 = 95_{10}$$

$$b. \quad abc_{16} = 10(16)^2 + 11(16) + 12 = 2748_{10}$$

Problem 17 Complete the binary addition table below. Then use it to perform the following long addition without switching to decimals.

+	0	1
0	0	1
1	0	10

	1	1	0	1	0	1
+	1	0	0	1	1	
1	0	0	1	0	0	0

Problem 18 Complete the trinary addition table below. Then use it to perform the following long addition without switching to decimals.

+	0	1	2
0	0	1	2
1	1	2	10
2	2	10	11

	1	2	0	2	2	1
+	1	1	2	1	2	
	2	0	2	2	1	0

Find the decimal representations of the trinary numbers you have used and check the answer.

$$\begin{aligned}
 120221_3 &= 3^5 + 2(3)^4 + 2(3)^2 + 2(3) + 1 \\
 &= 243 + 162 + 18 + 6 + 1 = 430_{10}
 \end{aligned}$$

$$\begin{aligned}
 11212_3 &= 3^4 + 3^3 + 2(3)^2 + 3 + 2 \\
 &= 81 + 27 + 18 + 3 + 2 = 131_{10}
 \end{aligned}$$

$$430_{10} + 131_{10} = 561_{10} = 202210_3$$

Problem 19 *Perform the following long subtraction of the ternary numbers without switching to the decimals.*

$$\begin{array}{r}
 1\ 2\ 0\ 2\ 2\ 1 \\
 -\ 1\ 1\ 2\ 1\ 2 \\
 \hline
 1\ 0\ 2\ 0\ 0\ 2
 \end{array}$$

Use the decimal representation found in Problem 18 to check your answer.

$$120221_3 = 430_{10}$$

$$11212_3 = 131_{10}$$

$$430_{10} - 131 - 10 = 299_{10} = 102002_3$$

Problem 20 Complete the octal addition table below.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

Then perform the following long addition of octal numbers without switching to the decimals.

$$\begin{array}{r}
 7602 \\
 + 455 \\
 \hline
 10257
 \end{array}$$

Find the decimal representations of the octal numbers you have used and check the answer.

$$7602_8 = 7(8)^3 + 6(8)^2 + 2 = 3970_{10}$$

$$455_8 = 4(8)^2 + 5(8) + 5 = 301_{10}$$

$$3970_{10} + 301_{10} = 4271_{10} = 10257_8$$

Problem 21 *An expression of the form $2 \times 3 = 10$ is written on the board. Does there exist a number system where it is correct? If you think that it does, find the following product in the system.*

$$5 \times 5 = 41_6$$