

FROM FRACTIONS TO DECIMALS (V2)

LAMC INTERMEDIATE GROUP - 10/06/13

WARM UP!

The instructors have been noticing some strange coincidences. Determine whether each of the following observations were indeed coincidences or if they were mathematically forced to happen.

- Derek had a magical bag filled with infinitely many marbles which were all either red, blue, green, yellow, or black. He tried to draw as long as he could without drawing two marbles of the same color. Derek was very disappointed when he noticed that he always finished in six turns or less.

- Deven and Morgan were looking at the roster for this class (including instructors!) and noticed that three people were born in the same month.

- As you can tell by looking at them, Pax and Cole each spend about 15 hours a day in the gym at UCLA. Pax told Cole, "I see you at the gym every day, what a strange coincidence"

- All the instructors were amazed to find out that two of them had the same favorite color of the rainbow!

- (1) Last week, and during presentations today, we showed that fractions with denominators whose only prime factors are 2 and 5 had terminating decimal equivalents. For the rest of the day our goal is to prove that all fractions have decimal equivalents that are **eventually periodic**. This means that the decimal eventually repeats itself over and over again, for example:

$$.00123123123\dots$$

is eventually periodic, because its eventually just repeated 123 over and over again.

- (a) Pax thinks this is a self-contradictory statement, claiming “If we proved that some fractions have terminating decimal equivalents, not all fractions can have eventually periodic decimal equivalents.” Is he right? Why or why not?

- (b) Despite Pax’s claim Derek thinks he has come up with a remarkably quick explanation of why every fraction has a decimal equivalent which is eventually periodic. He says, “There are only ten digits and once you use them all up you have to repeat yourself!” Is he right? Why or why not?

(c) Can you write down a few decimals that are not eventually periodic?

(2) Suppose we have two arbitrary fractions $\frac{a}{b}$ and $\frac{c}{d}$, $a, b, c, d > 0$.

(a) If

$$\frac{a}{b} = \frac{c}{d}$$

what can we say about the relationship between ad and cb ?

(b) What if

$$\frac{a}{b} < \frac{c}{d}?$$

(Hint: for both try some examples, say $\frac{1}{2} = \frac{3}{6}$ and $\frac{1}{3} < \frac{1}{2}$)

(3) Convert the following fractions into decimals and find their periods.

(a) $\frac{1}{2}$

(b) $\frac{1}{3}$

(c) $\frac{1}{4}$

(d) $\frac{1}{5}$

(e) $\frac{1}{6}$

(f) $\frac{1}{7}$

(g) $\frac{1}{8}$

(h) $\frac{1}{9}$

(4) In the previous problem, how did you know how many digits of $\frac{1}{7}$ to calculate?

(5) Based off all of your calculations in problem 3 can you guess what the graph of the function $f(x) = \frac{1}{x}$ looks like? Draw it here:

(6) Lets take a closer look at what we actually do when we preform long division to convert a fraction to a decimal. Since $\frac{3}{7}$ is Morgan's favorite fraction we'll try it with that.

- First we notice we can't actually divide 3 by 7 so we multiply by ten, then divide by 7 to find the first digit of the decimal representation. Which gives us 4 with a remainder of 2.

$$30 = 4 \cdot 7 + 2$$

which gives us that the first digit of the decimal is 4.

- The next thing we do is multiply the *remainder* of the previous step by 10 and again divide by 7.

$$20 = 2 \cdot 7 + 6$$

Hence the second digit of the decimal is 2.

- Find the next few terms in the decimal by writing out this process.

(7) Use the process from problem 6 to compute the decimal equivalent of $\frac{13}{15}$.

(8) Use the process from problem 6 to compute the decimal equivalent of $\frac{3}{22}$.

-
- (9) Here we finally prove what Derek thought he did at the beginning!
- (a) What (if anything) forces the process from problem 6 to repeat itself?
(*Hint: the next step in the process is determined by _____*)
- (b) What is the longest possible period length of a number of the form $\frac{a}{17}$?
- (c) What is the longest possible period length of a number of the form $\frac{a}{n}$?
- (d) Therefore we have shown that every fraction has a decimal equivalent which eventually repeats itself (and in fact we have shown more... we can control the length of the period based on the denominator (see (c)). [No need to write anything down for this part, I'm just summing it up]

(10) What is the 2013th digit after the decimal in the decimal equivalent of $\frac{1}{7}$?

(11) What is the 16^2 -th (nd? ...) digit after the decimal in the decimal equivalent of $\frac{1}{17}$?

HOMEWORK!

- (1) What is the maximum period length for the decimal equivalents of the following fractions:

(a) $\frac{5^4 3^8 9^{10} 11}{89^{10} 32^1 54^2}$

(b) $\frac{503 \cdot 671}{2013 \cdot 2012}$

- (2) What is the 2013th digit after the decimal place in the decimal equivalent of the following?

(a) $\frac{4}{7}$

(b) $\frac{503}{2000}$