“TAXICAB” GEOMETRY

MATH CIRCLE (HS1) 9/29/2013

It is a standard move in (planar) geometry to think of the points we are studying as being in the (x, y)-plane. We then have standard formulations of *points, lines,* and *angles*. We also have a way to measure distance, called the *Euclidean distance*, between the points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$:

$$d_E(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We'll call the geometry that arises using d_E *Euclidean geometry*.

What if we change this notion of distance? Suppose we instead define the *taxicab distance* as

$$d_T(P, Q) = |x_2 - x_1| + |y_2 - y_1|.$$

We'll call the geometry that arises using d_T (and keeping the same notions of points, lines and angles) *taxicab geometry.*

Note: Unless noted otherwise, all answers must be proven.

1) For the following P, Q: graph P and Q, calculate $d_E(P, Q)$ and $d_T(P, Q)$, and draw a shortest path from P to Q according to both d_E and d_T.

a) $P = (1, 1), Q = (4, 5)$

b) $P = (3, 2), Q = (1, 3)$

c) $P = (2, 3), Q = (-2, 3)$

d) $P = (-2, -2), Q = (2, 2)$

2) a) If $d_T(P, Q) = d_T(R, S)$, is it true that $d_E(P, Q) = d_E(R, S)$?

b) If $d_E(P, Q) = d_E(R, S)$, is it true that $d_T(P, Q) = d_T(R, S)$?

c) When is $d_E = d_T$?

d) Formulate and prove a relationship between d_E and d_T.

3) Let $P = (1, 1)$.

a) What geometric shape is $\{Q | d_E(P, Q) = 3\}$? Recall that $\{Q | d_E(P, Q) = 3\}$is the collection of points that are (Euclidean) distance 3 from P.

b) Come up with a suitable definition of a “taxicab circle”, i.e. how should we define a ‘circle’ in taxicab geometry?

c) Graph a taxicab circle with radius 3 and center P.

This handout is based on problems from previous math circle handouts as well as E. Krause “Taxicab Geometry.”

4) Let $\mathcal{C}$ be a (regular) circle with center $(0, 0)$ and radius 1. Give examples of circles that intersect it: nowhere, once, twice, and infinitely often.

5) Using Problem 4 as a starting place, examine how taxicab circles can intersect. How does this compare/contrast to circles?

6) Suppose you are given points P, Q. What does the set of points $\{R | d(P, R) + d(R, Q) = d(P, Q)\}$ look like for both $d = d_E$ and $d = d_T$? Start by looking at a few explicit examples.

7) For each set of points P, Q, graph $\{R | d(P, R) = d(Q, R)\}$ for both $d = d_E$ and $d = d_T$.

a) $P = (-2, 0), Q = (2, 0)$.

b) $P = (0, 0), Q = (4, 2)$.

c) $P = (-1, -1), Q = (2, 2)$.

d) Describe (in your own words) how to do this for arbitrary P, Q. Challenge: Prove your result analytically when $d = d_E$.

8) Let $P = (-3, 0)$ and $Q = (3, 3)$.

a) Graph $\{R | d_E(P, R) = 2 \cdot d_E(Q, R)\}$. What geometric shape do you get? Challenge: Prove this analytically!

b) Graph $\{R | d_T(P, R) = 2 \cdot d_T(Q, R)\}$. Compare this to the answer in a).

For Further Investigation (Homework):

1) Suppose we wanted to define an analogue of π in taxicab geometry. What might be a suitable value for "taxicab pi"? Discuss the advantages and disadvantages of your definition. Be sure to explore different approaches before coming to your conclusion!

2) Suppose we wanted to extend taxicab geometry into three dimensions. What might be a suitable way to measure distance? Explore the three dimensional analogues of Problems 3,6, and for a challenge, 7.