LAMC Junior Circle **June 9, 2013**

The year-end problem solving session solution manual

Problem 1

a. Solve the following equation. **5 pts**

$$x^2 \equiv 2 \ (mod\ 7)$$

x	0	1	2	3	4	5	6
x^2	0	1	4	9	16	25	36
$x^2\ (mod\ 7)$	0	1	4	2	2	4	1

$$x \equiv 3 \ (mod\ 7) \text{ or } 4 \ (mod\ 7)$$

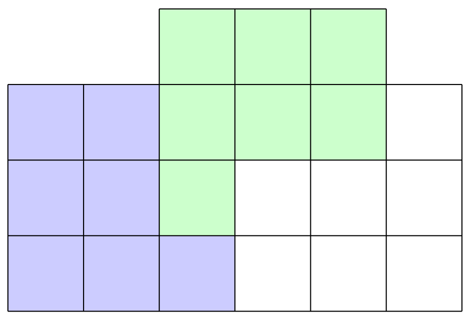
b. **5 pts**

$$\sqrt[3]{3} \equiv \qquad\qquad (mod\ 5)$$

This is equivalent to solving the equation $x^3 \equiv 3\ (mod\ 5)$. Do as above.

Problem 2 **10 pts**

Trace the lines that would cut the following figure into three parts of equal shape and size.



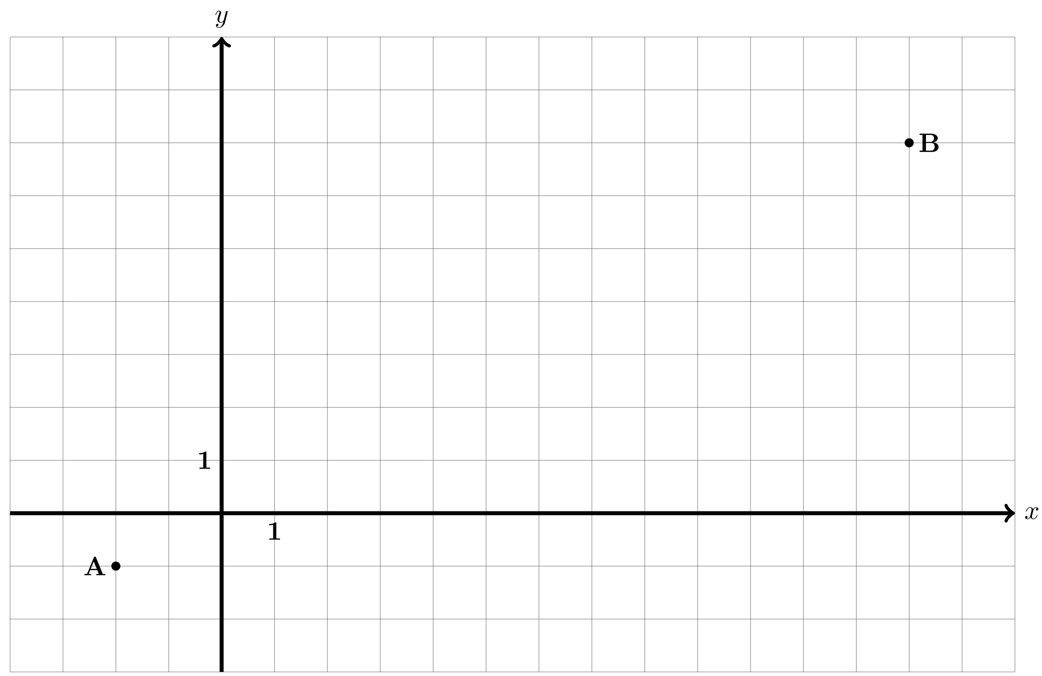
Problem 3 **10 pts**

Find the number that is seven times greater than its last digit.

$$35 = 7 \times 5$$

Problem 4 **10 pts**

The distance $|AB|$ is a whole number. Use the Pythagoras' theorem to find it.



$$|AB|^2 = 15^2 + 8^2 = 225 + 64 = 289$$

$$|AB| = 17$$

Problem 5 **10 pts**

Write down the first six lines of the Pascal's triangle in the space below.

$n = 0$:						1					
$n = 1$:					1		1				
$n = 2$:				1		2		1			
$n = 3$:			1		3		3		1		
$n = 4$:		1		4		6		4		1	
$n = 5$:	1		5		10		10		5		1

Problem 6 **10 pts**

Use the previous problem to expand the following product.

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

It's not the answer, but the way of thinking that matters here.

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$$

The expansion of the sum is a bunch of monomials. To construct one monomial, you have to pick a variable, either an x or a y, from each of the five parenthesis. To assemble x^5 this way, you need to pick no y's. There is only one way to pick no objects out of five.

$$\binom{5}{0} = 1$$

Note that 1 happens to be the first entry of the last line of the previous problem.

To construct the x^4y monomial, you need to pick only one y out of the possible five. There are five different ways to do it.

$$\binom{5}{1} = 5$$

This happens to be the second entry of the last line of the previous problem. Thus, the corresponding monomial is $5x^4y$.

To construct the x^3y^2 monomial, you need to pick two y's out of the possible five.

$$\binom{5}{2} = 10$$

This happens to be the third entry of the last line of the previous problem. Thus, the corresponding monomial is $10x^3y^2$. And so on.

Problem 7 **10 pts**

You toss a fair coin five times. How likely are you to get either two heads or two tails?

Once again, it's the way of thinking that matters. Let x be the probability of getting a head in a single toss and let y be the chance of getting a tail. Naturally, $x + y = 1$. The tosses are independent, so

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y).$$

The expansion

$$(x+y)^5 = 1 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

gives you the probabilities of all the possible outcomes at once. For example,

$$P(\text{two heads}) = 10x^2y^3 = 10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{10}{2^5} = \frac{5}{16}$$

Similarly

$$P(\text{two tails}) = 10x^3y^2 = 10\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 = \frac{10}{2^5} = \frac{5}{16}$$

Finally, here comes the answer.

$$P(\text{two heads or two tails}) = \frac{5}{16} + \frac{5}{16} = \frac{5}{8}$$

Problem 8 **10 pts**

There are 23 students, 3 of them identical (indistinguishable) triplets, in a class. How many ways are there to put the class in one line for taking pictures in such a way that the triplets are in the middle of the line? Answering this question, do not compute the factorials!

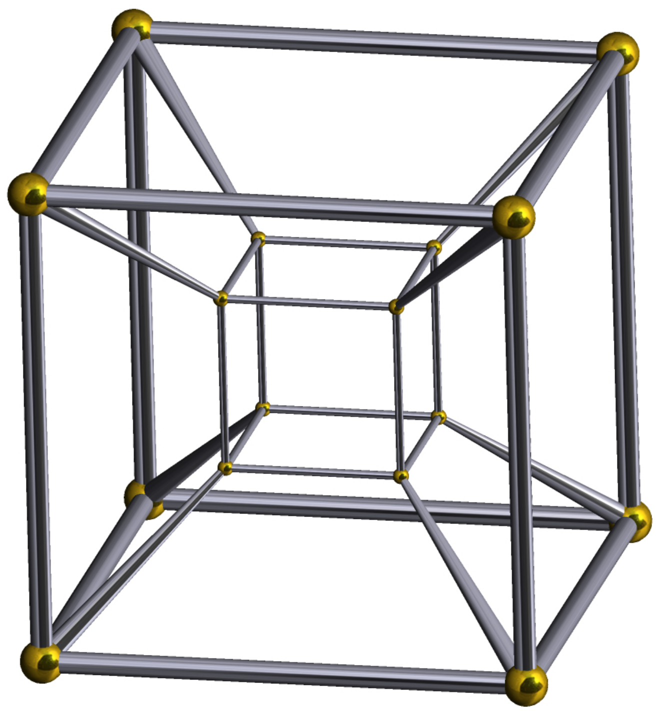
Since the triplets are indistinguishable, there is only one way to put them next to one another in the center. There remain 20 more distinguishable students, so the answer is

$$20!$$

(Reads as 20 factorial, not as 20 exclamation mark.)

Problem 9 **10 pts**

Draw a 4D cube (a.k.a. a hypercube and a tesseract) in the space below. How many 3D faces does it have?



Taking a good look at the picture produces the answer.

The number of the 3D faces = 8.

Problem 10 **10 pts**

Hiking in Yosemite, Oleg used a rectangular soap bar to wash the dishes, spending the same amount of soap each time. After seven washes, the length, width, and height of the soap bar became two times smaller than the original sizes. How many more times can Oleg wash the dishes with the soap?

$2^3 = 8$, hence there is enough soap for one more chore. Time to go home!