Geometry when you only know distances

June 7, 2013

1 Notation

For a set A, we will write $x \in A$ to mean ”x is in A’.” For example, $1 \in \{1,2,3\}$. We’ll also write $A \subseteq B$ to say that ”A is contained in B.” For example, $\{1,2,3\} \subseteq \{1,2,3,4\}$.

We will also use the *infimum* of a set: if $A \subseteq \mathbb{R}$, we define $\inf A$ to be the biggest lower bound of A. So for example, if $A = \{t : 1 < t < 2\}$, then the numbers 0,1,-1 are all lower bounds for this set, but 1 is the biggest of them all. Note that the infimum of a set doesn’t have to be inside the set (in this case, 1 is not in the set A).

2 Metric Spaces

A *metric space* is a set X equipped with a function d that takes pairs of points in X as input such that for any $x, y, z \in X$,

a $d(x,y) \geq 0$

b $d(x,y) = d(y,x)$

c $d(x,y) = 0$ if and only if $x = y$

d $d(x,y) \leq d(x,z) + d(z,y)$.

Determine whether the following sets with their prescribed distances are metric spaces.

1. (The Los Angeles Metric) $X = \mathbb{R}^2 \backslash \{0\}$, and $d(x,y) = 0$ if $x = y$ and 1 if $x \neq y$.
2. $X = \mathbb{R}^2_+ = \{(x_1, x_2) : x_2 > 0\}$, $d(x,y) = |x_1 - y_1| + \left|\log \frac{x_2}{y_2}\right|$.
3. Let $X = \mathbb{R} \cup \{\infty\}$ and define $d(x,y) = |x - y|$ for $x, y \in \mathbb{R}$ and $d(x, \infty) = 1$, and $d(\infty, \infty) = 0$.
4. Let $X = \mathbb{R}$, $n \in \mathbb{N}$, and $d(x,y) = |x-y|^{\frac{1}{n}}$.

5. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|^2$.

6. (Hamming Metric) Fix n and let X be the set of strings of 0's and 1's of length n (so for example, if $n = 2$, then $X = \{(0,0), (1,0), (0,1), (1,1)\}$ and if $x, y \in X$, define $d(x, y)$ to be the number of positions in the strings x and y where they disagree. For example, $d((01001), (01110)) = 3$.

7. Let $G = (V, E)$ be a graph with edges E and vertices V. Let $X = V$ and let $\rho(x, y)$ be the length of the smallest path connecting x and y, that is, the smallest amount of edges in a path traveling in G from x to y.

8. Let X be the set of second degree polynomials on $\mathbb{R}$, and for two polynomials p and q, define $d(p, q) = |p(0) - q(0)| + |p(1) - q(1)|$.

9. Let X be the set of second degree polynomials on $\mathbb{R}$, and for two polynomials p and q, define $d(p, q) = |p(0) - q(0)| + |p(1) - q(1)| + |p(2) - q(2)|$.

10. If X is a metric space, does every point have a midpoint? That is, if $x, y \in X$, is there always $m \in X$ so that $d(x, m) = d(m, y) = \frac{d(x,y)}{2}$.

3 Sub-metric spaces

If $X \subseteq \mathbb{R}^2$, and $d(x, y) = |x - y|$ for all $x, y \in X$ (here, $|\cdot|$ is the usual Euclidean distance), then X is a metric space. Then this is a metric space, since all the properties of being a metric are satisfied by d already since they hold for points in $\mathbb{R}^2$, so they will also hold for points in X.

This leads to an interesting question: which metric spaces can be realized as subsets of Euclidean space? In other words, let's say I have a metric space $X = \{x_1, ..., x_n\}$ with metric d and I want to pick $x_1, ..., x_n$ to lie in some $\mathbb{R}^d$ so that $d(x_i, x_j) = |x_i - x_j|$. When is this possible?

For example, if I have a metric space $X = \{x_1, x_2, x_3\}$ and $d(x_1, x_2) = d(x_1, x_3) = 1$ and $d(x_2, x_3) = \sqrt{2}$, I can pick x_1, x_2, x_3 to be the vertices of a right triangle in $\mathbb{R}^2$ with side lengths $1, 1, \sqrt{2}$.

Let's consider a few cases.

1. Can you find 2 points x_1 and x_2 in some $\mathbb{R}^d$ (with the Euclidean metric) so that the distances are whatever you want?

2. Ok, but now what about a three point metric space? Suppose $\{x_1, x_2, x_3\}$ is a metric space with metric d and I want to pick the x's to be in $\mathbb{R}^2$ so that $d(x_i, x_j)$ actually equals $|x_i - x_j|$. Can I do this?

3. How about 4 points in some $\mathbb{R}^d$? (The answer is no! There are particular distances you can prescribe so that you cannot find in *any* $\mathbb{R}^d$ points x_1, x_2, x_3, x_4 with those distances.)

4 Path Metrics

Let X be any subset of $\mathbb{R}^d$. We say X is *connected* if any two points x and y may be joined by a path of finite length. We define the *path metric* $\rho(x, y)$ is equal to the infimum of the lengths of all paths in X connecting x and y.

1. Show that ρ is a metric.
2. What is $\rho(x, y)$ if there is no path connecting x and y?
3. If X is connected, and $x, y \in X$, is there always a path γ connecting x and y so that the length of γ equals $\rho(x, y)$?
4. If X is connected, and $x, y \in X$, and there is a path γ connecting x and y whose length is $\rho(x, y)$, is it unique?

5 Quasiconvex spaces

A metric space X is C-quasiconvex if the path metric ρ satisfies $\rho(x, y) \leq Cd(x, y)$, that is, every pair of points in X may be connected by a path.

1. Show that if $X \subseteq \mathbb{R}^2$ and is C-quasiconvex, then $C \geq 1$.
2. Let X be the boundary of a rectangle with dimensions 1 and $L > 1$. What is the smallest C so that X is C-quasiconvex?
3. Let X be the unit circle, equipped with the Euclidean metric (so $d(x, y) = |x - y|$), and let ρ be its path metric. What is the smallest C so that X is C-quasiconvex?
4. Can you think of a connected set that is not C-quasiconvex for any $C > 1$?
5. (Hard, but fun!) Let $X \subseteq \mathbb{R}^2$ be any bounded set that divides $\mathbb{R}^2$ into at least two pieces, one of which is bounded and contains a ball. Show that if X is C-quasiconvex, then $C \geq \frac{2\pi}{3\sqrt{3}}$.