PIGEONHOLE PRINCIPLE

BEGINNER CIRCLE 6/2/2013

1. The Sock Drawers

Problem 1. Everyday before he goes to school, Jeff draws socks. Unfortunately, as school starts at 5AM, it is always too dark to see what color of sock he is drawing. If there are 2 different colors of socks that he can draw, how many socks must Jeff draw before he has for certain drawn 2 socks of the same color?

Problem 2. As Isaac is more fashionable than Jeff is, he has 7 different colors of sock to draw. How many socks must Isaac draw before he has drawn 2 socks of the same color?

Problem 3. When Jonathan studied in the perpetually dark Himalayas, he used to draw both socks and mittens. When Jonathan draws clothes, he always draws one sock, and one mitten. Unfortunately, he cannot see which of the 3 colors of socks or mittens he has drawn. How many times must he draw socks and mittens before he for certain has mittens that match, and socks that match.

Problem 4. What if Jonathan wanted his socks and mittens to match?

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Problem 5. Derek decides as a gift to his 4 math circle friends, he will draw them socks. Derek has 3 different colors of sock to draw; why do 2 of his friends get socks of the same color as a gift?

Problem 6. Moran has not only socks and mittens, but also a pair of shoes that he wears. He demands that his socks match each other, his mittens match each other, and that his shoe match. He has 4 different colors of socks, shoes and mittens, and whenever he draws, he draws 1 sock, 1 shoe and 1 mitten. How many times must he draw clothes in the morning?

2. The Pigeonhole Principle

The Pigeonhole principle says “If you have n holes to stuff pigeons into, and you have more than n pigeons, then at least 1 hole has at least 2 pigeons in it.” For each problem, identify what the “pigeons” are and what the “holes” are.

Problem 7. Why are two people in this class born on the same day of the week?

Problem 8. Suppose that nobody has more than 1 million hairs on their head. Why is it that there are two people in the world that have exactly the same number of hairs on their head?

Problem 9. Suppose that we have a party with 33 people, and people shake hands with eachother. Why is that two people necessarily shook the same number of hands?

Problem 10. Suppose that you have picked 21 whole numbers. Show that there are two numbers in this set whose difference is divisible by 20. (Use modular arithmetic!)

Problem 11. Suppose that 6 people are at a party, and shake hands with eachother. Show that there are either 3 people who all shook hands, or 3 people who did not all shake hands. (Hint: First show that there is a person who either shook hands with 3 different people, or did not shake hands with 3 people)

Problem 12. Suppose that you throw 17 darts at a 4cm by 4 cm dartboard. Show that two of the darts are within a distance of 2cm of eachother.

Problem 13. Suppose we take a grid and color the intersections red and blue. Show that you can draw a rectangle on the grid whose four corners are all the same color.

3. The Strong Pigeonhole Principle

The Strong Pigeonhole principle says "If you have n holes and k pigeons that you are stuffing into holes, then there is hole that contains at least the average number of pigeons (rounded up)"

Problem 14. Billiards is played with 15 cue balls, and a table with 6 pockets. Why is there necessarily a pocket that will have 3 balls in it at the end of the game?

Problem 15. Why are there at least 6000 people in the world that have the same number of hairs on their head?

Problem 16. In order to become a master mathematician in 30 days, you will have to solve math problems for exactly 45 minutes, and work on math for at least 1 minute a day. Show that there is a consecutive period of days where you do exactly 14 minutes of math. (Use modular Arithmetic!)

Problem 17. On an 8 by 8 chessboard, 17 rooks are placed are placed. Show that there are at least 3 rooks that do not threaten each other. (Hint: Can you show that one row contains at least 3 rooks, and a different row contains at least 2 rooks, and still have at least 1 rook remaining?)

Problem 18. Suppose I have a sequence of 10 different numbers, like

$$3, 7, 5, 6, 9, 8, 0, 1, 2, 4$$

Let $f(k)$ be the length of the longest increasing subsequence that starts at the kth spot. For instance, the longest increasing subsequence that starts at the 1st spot is

$$3, 5, 6, 8$$

so $f(1) = 4$.

(1) For this particular sequence of numbers, what is $f(2)$? What about f(6)?

(2) Now we suppose that we have some different sequence of 10 numbers. Suppose that there is no increasing subsequence of length 4 or greater. Show that there are at four different k such that $f(k)$ takes the same value.

(3) Why does this show that there is a decreasing subsequence of length of at least length 4?

(4) Conclude that for every sequence of 10 different numbers, there is always an increasing or decreasing subsequence of length 4 or longer.

Problem 19 (Baby Dilworth)**.** Using the above argument show that every sequence of n^2+1 different numbers contains an increasing or decreasing subsequence of length at least $n+1$.

Problem 20. You start coloring a piece of paper red, yellow and blue, until every single part of the paper has been colored. (You are allowed to do tricky things, like draw infinitely detailed pictures, use very thin lines, create infinitely small dots in your drawing). Show that you can find a rectangle whose four corners are all the same color.