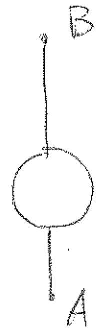
UCLA MATH
CIRCLE
05/19/2013

How to get from point A to point B.

Ex 1



Infinite # of ways
(since we can go around the circle ∞ly many times).

Definition Let X be a space.

A path in X from A to B is a function $\gamma(t)_{0 \le t \le 1}$, such that

(1) $\gamma(t)$ is continuous (no jumps)

(2) $\gamma(0) = A$, $\gamma(1) = B$.

Exercise: If there is one path from A to B, then there are infinitely many paths from A to B.

For example, for any number $n \in \mathbb{N}$, look at the path
$$\gamma_n(t) := \gamma(t^n)$$

Also, one can consider
$$\tilde{\gamma}(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ B & \frac{1}{2} \le t \le 1 \end{cases}$$

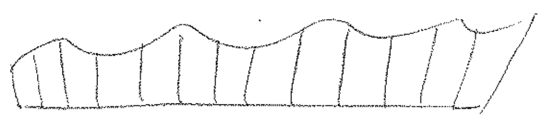
To study our question: "How many ways are there to get from A to B?"

We need an equivalence relation

Possible conditions:
changing speeds should not make any difference.

e.g. $t \longmapsto t^n$

- only hit each pt once. (This a restriction).
- Two paths can be connected point by point without jumps in the middle

e.g. 

Definition: Let $\gamma(t)$ and $\beta(t)$ be paths from A to B. A homotopy from γ to β is a continuous function $H(s,t)$ such that $0 \le s,t \le 1$ and

(1) $H(0,t) = \gamma(t)$

$H(1,t) = \beta(t)$

(2) $H(s,0) = A$

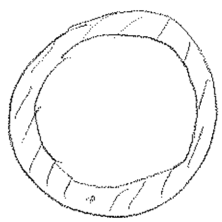
$H(s,1) = B$.

Definition: Two paths γ & β from A to B are homotopic if there exists a homotopy H from γ to β.

We will write $[\gamma] = [\beta]$.

Homework Exercise

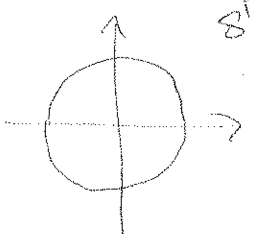
Let $\gamma(t)$ be a path from A to B, and
$\beta(t) = \gamma(t^2)$.
Show that β and γ are homotopic.



A

$\parallel$

$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$



(1) Stand on A: $\gamma(t) = A$

(2) Travel once ~~circle~~ around circle in the counterclockwise ~~ly~~ direction

$$\gamma(t) = e^{2\pi i t}$$

(3) Once around clockwise

$$\gamma(t) = e^{-2\pi i t}$$

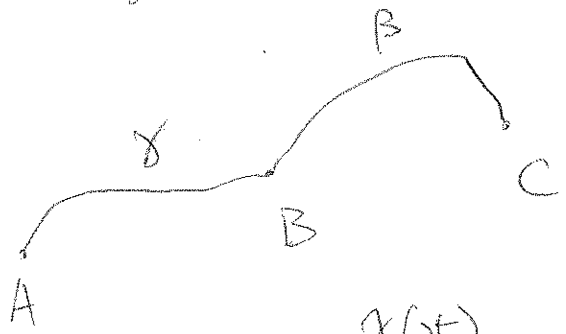
(4) Go around n times

$$\gamma(t) = e^{2\pi i n t}, \quad n \in \mathbb{Z}.$$

Q: Are these paths different, i.e. not homotopic?

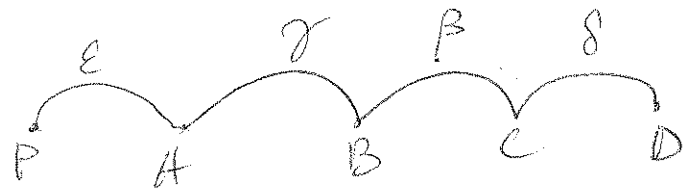
(1) and (2) are not homotopic

The Algebraic Structure



Warning
We can only multiply two paths if one starts at where the other ends.

define a path
$$(\gamma * \beta)(t) = \begin{cases} \gamma(2t), & 0 \le t \le \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$



We want:

- associative

$$[(\delta * \beta) * \gamma] = [\delta * (\beta * \gamma)]$$

- For each A, a path c_A from A to A such that

$$[c_A * \varepsilon] = [\varepsilon]$$
$$[\gamma * c_A] = [\gamma].$$

$$c_A(t) = A \quad \text{for all } t.$$

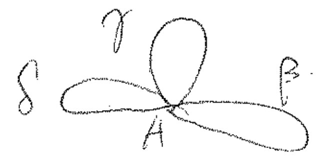
- Commutative not applicable in this case.

- Inverse

$$\gamma^{-1}(t) = \gamma(1-t)$$

Fact: $[\gamma * \gamma^{-1}] = [c_A]$

Same endpoints, i.e. paths from A to A.



(1) $[(\gamma * \beta) * \delta] = [\gamma * (\beta * \delta)]$

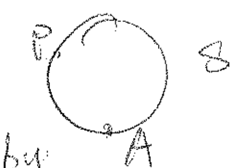
(2) $[\gamma * c_A] = [\gamma]$

(3) $[\gamma * \gamma^{-1}] = [c_A] = [\gamma^{-1} * \gamma]$.

This is similar to $\mathbb{R} - \{0\}$ with $*$ replaced by multiplication.

or $\left\{ \begin{array}{c} \text{invertible functions} \\ f : \mathbb{R} \to \mathbb{R} \end{array} \right\}$ with $*$ replaced by composition.

e.g. $f(x) = x$



We denote the (set of) loops @ A up to homotopy by

$\pi_1(S^1, A)$.

Function $f : \pi_1(S^1, A) \longrightarrow \mathbb{Z}$ ← integers.

$\gamma \longmapsto$ # of times γ passes P counterclockwise

− # of times γ passes P clockwise.

Properties

- $f(\gamma * \delta) = f(\gamma) + f(\delta)$.
- $f(c_A) = 0$
- $f(\gamma^{-1}) = -f(\gamma)$.
- f independent of P.

Fact: $[\gamma] = [\delta] \iff f(\gamma) = f(\delta)$

So $\pi_1(S^1, A) = \mathbb{Z}$.