PI ARE SQUARE

(But I thought they were round!)

Math Circle (Intermediate) 05/12/13

What is π?

The ancients observed that the circumference L of a circle increases if you increase the diameter d. Moreover, if you change the diameter by a certain factor (i.e., replace d by $k \cdot d$ for some $k > 0$), the circumference changes in the same way (L is replaced by $k \cdot L$). Thus the ratio

$$\frac{L}{r}$$

is constant. The number π is defined as the numerical value of this ratio.

With this notation, the formula for the circumference (length of a circle) is

$$L = \pi \cdot d_____,$$

where r is the radius.

From circumference to area

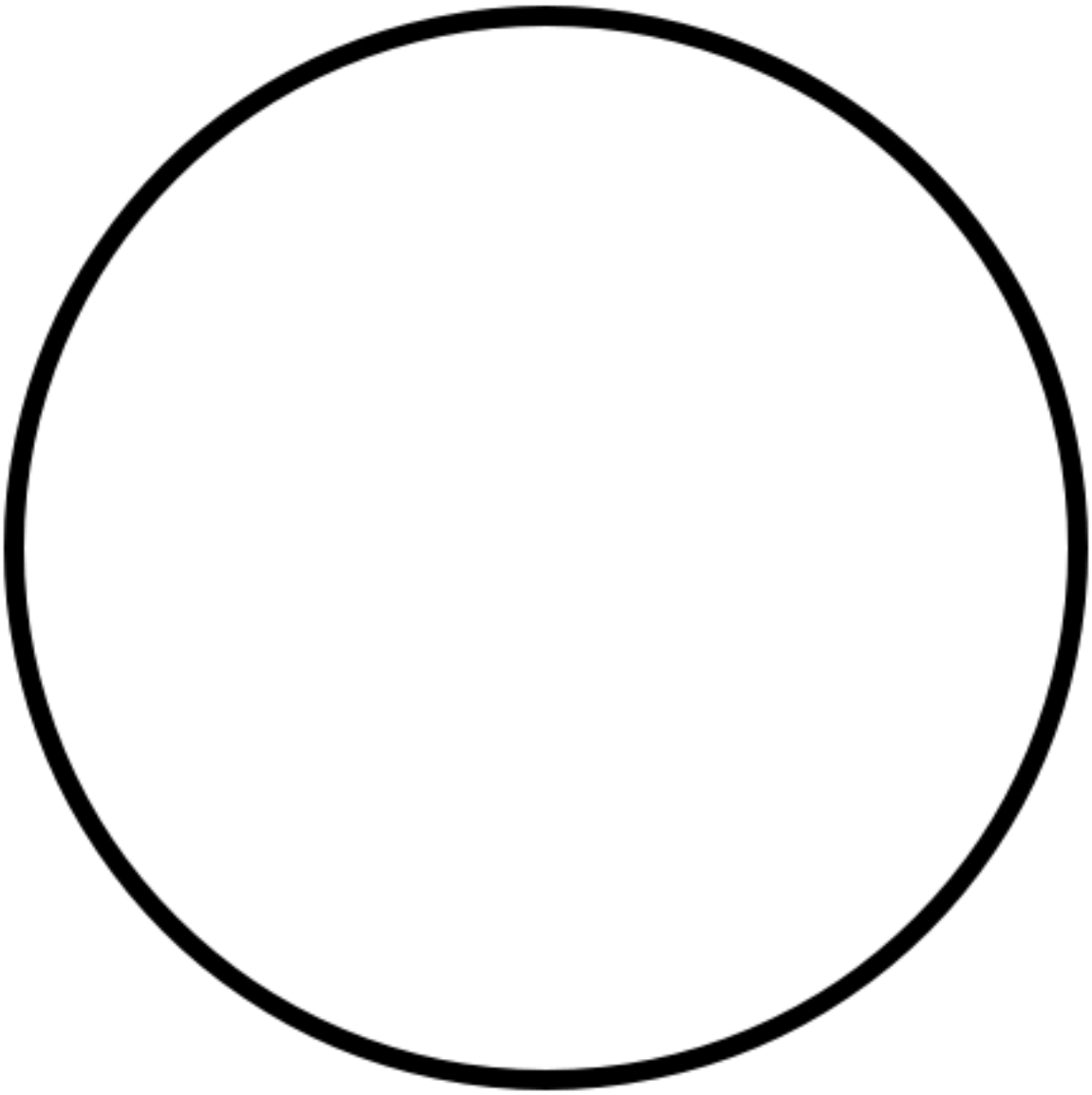
In the same way, one can notice that the area S inside of a circle is proportional to the *square* of its radius, r^2. Do we need a new constant representing the value of $\frac{S}{r^2}$, or is this constant related to π in some way?

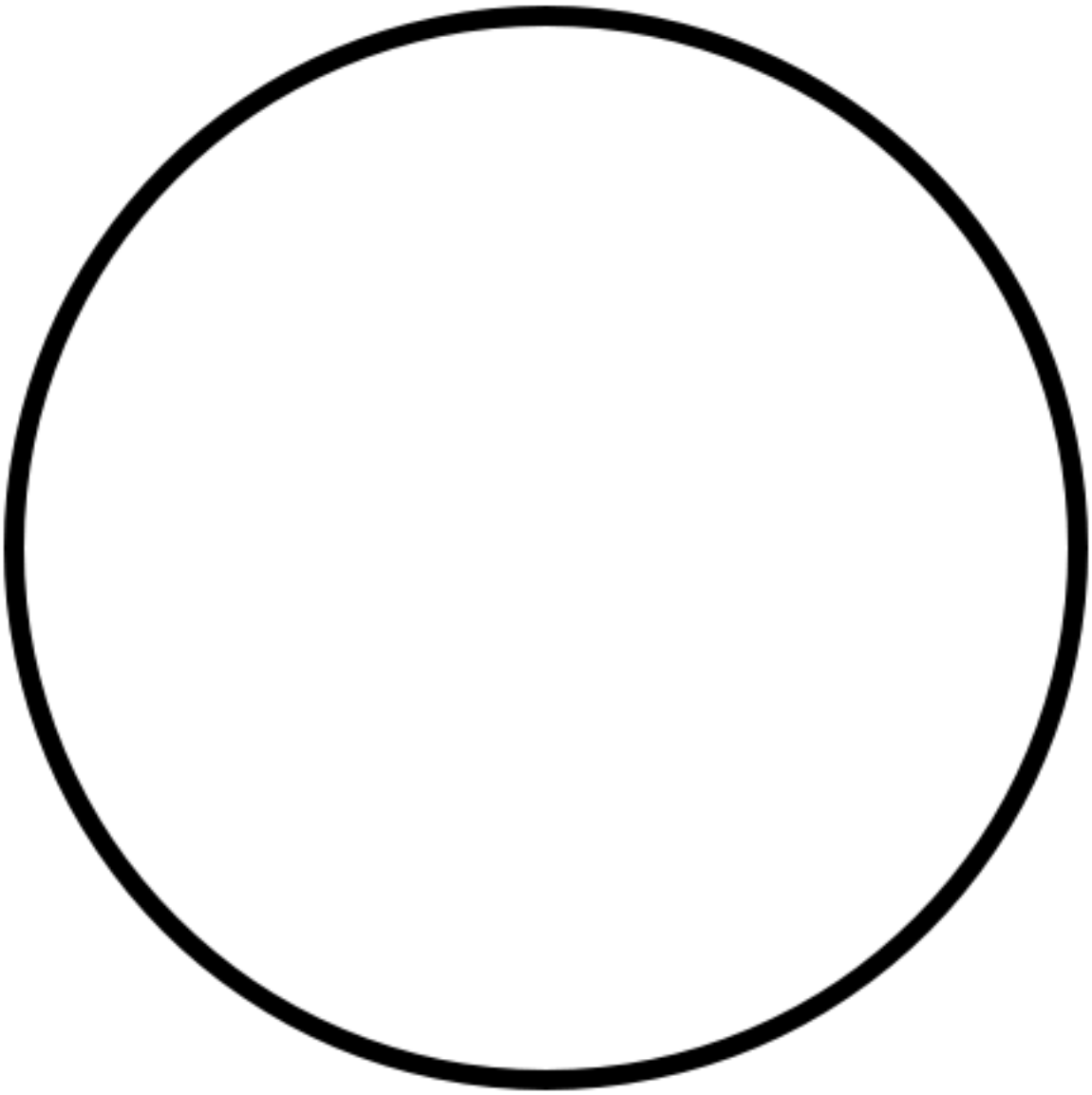
Here is an argument that allows one to demonstrate the formula for the area.

- divide the inside of the circle into several equal wedges (sectors) for which
  - one of the vertices of each wedge is the center;
  - the two sides adjacent to the center are radii;
  - the third side is an arc of the circle;

- now cut the picture along the straight sides of the triangles;
- rearrange the triangles so that all together they form a "parallellogram-like" shape;
- imagine that the number of parallelograms gets bigger and bigger so that the triangles become more and more narrow;
  - what happens with the curved sides of the triangles?
  - what happens to the "parallelogram-like" shape?
- Suppose that we make the subdivision of the inside of the circle into the triangles finer and finer and will keep rearranging the triangles into a big parallelogram.
- What is the height of the parallelogram (in terms of the radius)?
- What is the length of the longer side of the parallelogram (in terms of the circle)?
  - What is the area of the parallelogram?
- Thus, the area of the inside of a circle of radius r is

$$S = \qquad\qquad ;$$





Let's take a look at inscribing polygons in circles. Remember, a polygon is **inscribed** in a circle if all of its vertices lie on the circle.

As you proved on last week's handout, any polygon can be inscribed in a circle so long as it is _____.

For the following problems, express your answer as (a) an exact value (perhaps involving radicals), and (b) a numerical approximation to 3 decimal places (you may use a calculator for this).

1. Find the perimeter of an equilateral triangle inscribed in a unit circle.

2. Find the perimeter of a square inscribed in a unit circle.

3. Find the perimeter of a regular hexagon inscribed in a unit circle.

For the following problems, express your answer as a numerical approximation to 5 decimal places (you may use a calculator). See the hints on the board.

4. Find the perimeter of a regular octagon inscribed in a unit circle.

5. Find the perimeter of a regular 50-sided polygon inscribed in a unit circle.

6. Find the perimeter of a regular 500-sided polygon inscribed in a unit circle.

7. Find the perimeter of a regular 2000-sided polygon inscribed in a unit circle.

8. What value are your answers for Problems 1-7 approaching as you increase the number of sides? **Why do you think this is?**

More problems:

9. Find the area of a sector with angle α cut out of disk of radius r.

10. Find the ratio of the area of a sector intercepting a 60^0 arc to the area of the disk inscribed into this sector.

11. Compute the area of a ring if the chord of the outer boundary circle tangent to the inner boundary circle has length 2.

    (a) Is there a unique ring that satisfies this condition? If not, how many such rings can you find?

    (b) What can you say about the possible values of the *outer* radius of the ring?