

Divisibility by 11

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Let $x = \overline{x_1x_2\dots x_{n-1}x_n}$ be a representation of the positive integral number x in the decimal place-value system. In other words,

$$x = x_n + 10x_{n-1} + 100x_{n-2} + \dots + 10^{n-1}x_1 \quad (1)$$

where x_i are some decimal digits, the integral numbers zero through nine. Since a number cannot begin with zero, its first digit, x_1 cannot be equal to zero either. For example, $x = 12345$ means that $x = 5 + 4 \times 10 + 3 \times 10^2 + 2 \times 10^3 + 1 \times 10^4$.

The purpose of this note is to prove the following.

Theorem 1 *The number x is divisible by eleven if and only if the alternating sum of its digits is divisible by 11.*

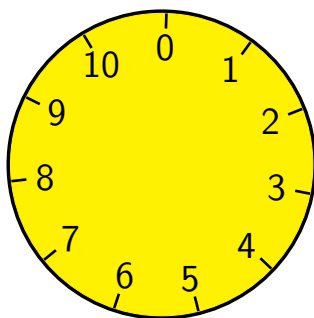
$$(x_1 - x_2 + x_3 - \dots + (-1)^{n-1}x_n) \mid 11 \Leftrightarrow x \mid 11 \quad (2)$$

(The \mid sign reads "is divisible by". The sign \Leftrightarrow reads "if and only if".)

The below proof was worked out by one of our students, Ben Volokh, with the help of his father, Eugene Volokh. Oleg Gleizer

has added some minor details and explanations.

Recall that the $\text{mod } 11$ arithmetic is the arithmetic of the face of an 11-hour clock.



In this arithmetic, $11 \equiv 0 \pmod{11}$. (We reserve the standard $=$ sign for the number line.) It is also the arithmetic of the remainders for the division by 11. We will need the following facts.

$10 \equiv -1 \pmod{11}$. Take one step in the counter-clock-wise direction from 0 and you will end up at 10.

$100 \equiv 1 \pmod{11}$. Indeed, $100 = 10 \times 10 \equiv (-1) \times (-1) \equiv 1 \pmod{11}$.

$1000 \equiv -1 \pmod{11}$. Indeed, $10^3 \equiv (-1)^3 \equiv -1 \pmod{11}$.

In general, $10^n \equiv (-1)^n \pmod{11}$.

Proof of Theorem 1— If x is divisible by 11, then $x \equiv 0 \pmod{11}$. Thus $x_n + 10x_{n-1} + 100x_{n-2} + \dots + 10^{n-1}x_1 \equiv 0 \pmod{11}$. But $x_n + 10x_{n-1} + 100x_{n-2} + \dots + 10^{n-1}x_1 \equiv x_n - x_{n-1} + x_{n-2} - \dots + (-1)^{n-1}x_1 \equiv 0 \pmod{11}$. Which means that the alternating

sum of the digits of the number x , $x_n - x_{n-1} + x_{n-2} - \dots$, is divisible by 11. The argument can be easily reversed. \square