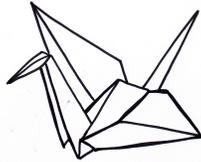


# GEOMETRY 1

BEGINNER CIRCLE 5/5/2013

## 1. MAKING AN EQUILATERAL TRIANGLE.

Origami is a funny thing. You may be familiar with a compass and straight edge, the tools that the ancient Greeks used to do geometry. The Japanese, on the other hand, used *origami*, which is a paper-folding art. You've probably seen some examples of origami from the art point of view



However, you can also use Origami to do geometry!

**Problem 1.** Can you find a line that goes through points  $a$  and  $b$ ?

$b$   
●

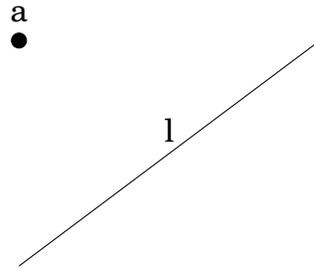
$a$   
●

**Problem 2.** Find the point that lies exactly halfway between these two points?

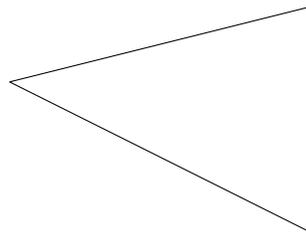
$a$   
●

$b$   
●

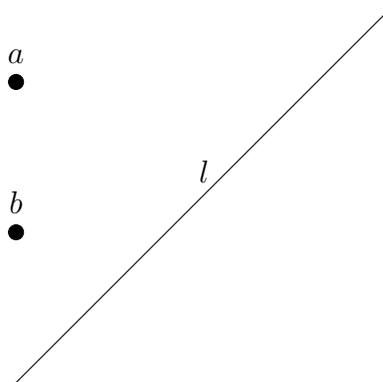
**Problem 3.** Can you find a line that has a right angle with the line  $l$  and goes through the point  $a$ ?



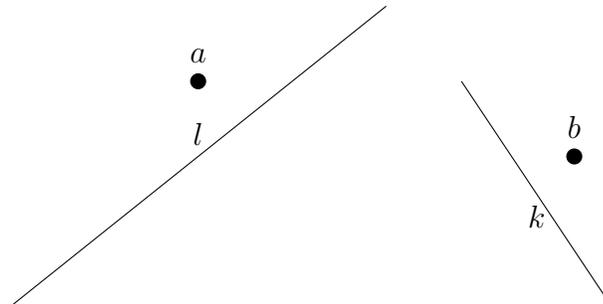
**Problem 4.** Given the angle below, can you find a line that cuts it in half?



**Problem 5.** Can you find fold  $a$  onto line  $l$  with a fold that goes through  $b$ ?



**Problem 6.** Can you find a fold that takes points  $a$  onto line  $l$ , and point  $b$  onto line  $k$ ?

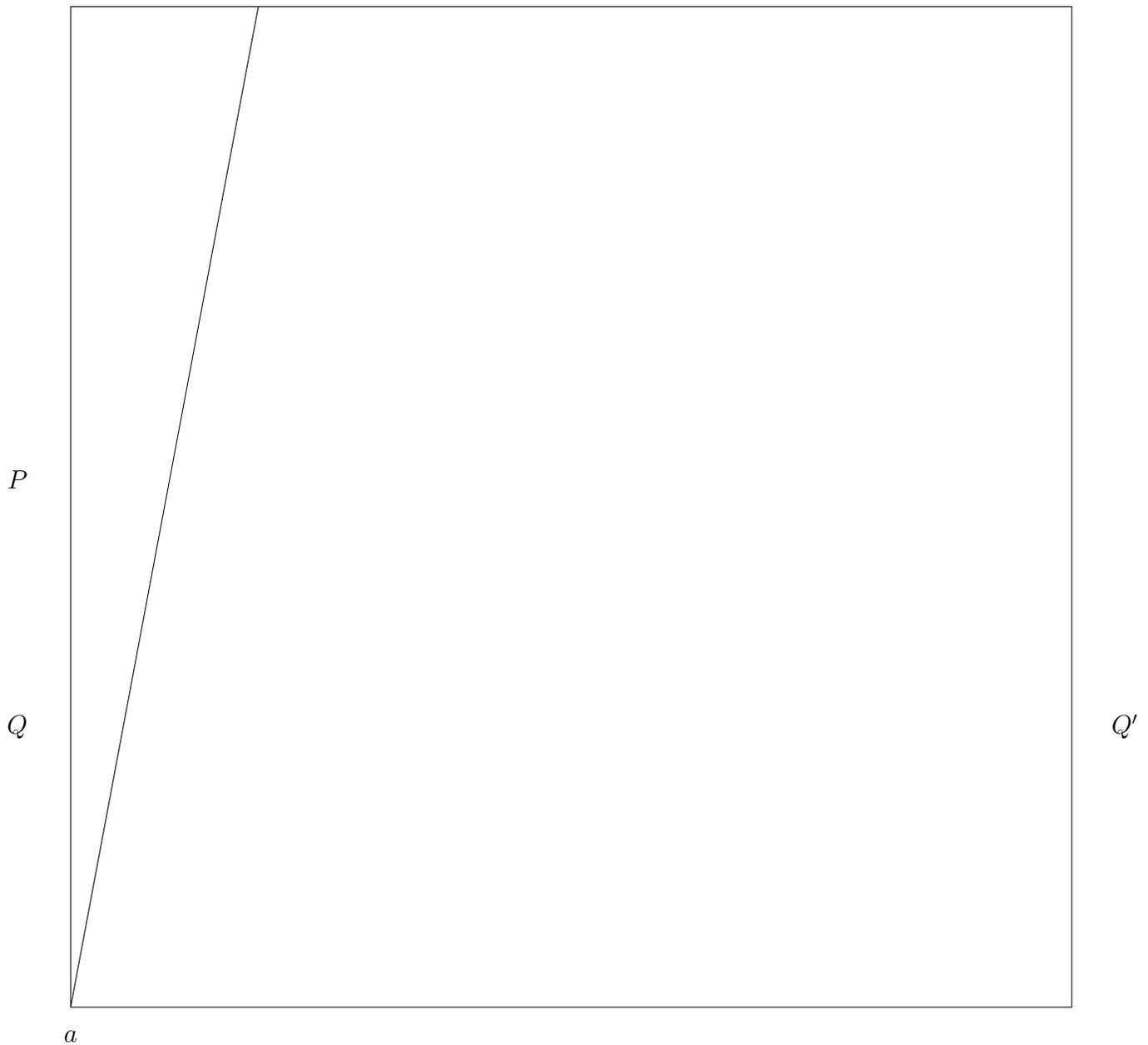


**Problem 7.** An equilateral triangle is a triangle whose side lengths are all the same.

- (a) One fact that you can use when folding the equilateral triangle is that it is symmetric, meaning that the left side is the same as the right side. Start by drawing a line to represent one of the three sides of the triangle.
- (b) Label the two endpoints of the line  $A$  and  $B$ . These represent the first 2 corners of your triangle.
- (c) Fold  $A$  to  $B$  (Call this fold  $F$ ). Why does the third corner of the triangle have to lie somewhere on the fold that you just made?
- (d) Make a fold that goes through  $A$ , taking the point  $B$  onto the fold  $F$ . Mark the point where  $B$  ends up on  $F$ . What should this point represent?
- (e) Make an equilateral triangle.

**Problem 8.** In this problem we trisect an angle.

- (a) Take the square containing the angle, and fold it in half 2 times length wise, and two times width wise, and unfold. You should get 16 smaller squares.
- (b) Fold the point  $A$  to the first horizontal fold (Labeled with  $Q'$ ) in such a way that the point  $P$  lands on the line labeled with a  $C$  on the end.
- (c) Mark where the point  $Q$  wound up after the fold with an  $x$ . Mark where the point  $P$  wound up after the fold with a  $y$ . Then unfold your paper.
- (d) Draw a line from  $x$  to  $A$  and from  $y$  to  $A$ . Congrats! You've trisected the angle drawn in the square!



## 2. KIRIGAMI AND AREA

What is area? Area is given by the number of squares that you can fit inside another shape. For instance, we say that a square that has a side length of 1 cm has an area of  $1\text{cm}^2$ .

Here is the (slightly unusual) definition of area that we will use

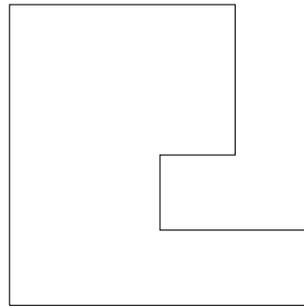
**Definition 1.** The **area** of a rectangle that is 1 cm high and  $x$  cm long is  $x$ .

From here, we will discover how to get all different kinds of areas for different shapes. The **scissor cutting principle** says that if you take a shape, cut it up into a bunch of smaller shapes, and rearrange them so that they don't overlap, that the new shape has the same area as the old one.

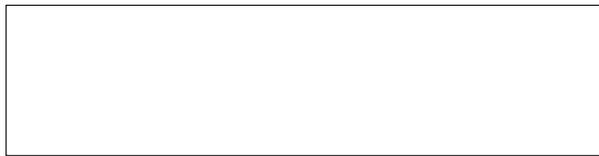
**Problem 9.** Using the scissor cutting principle, find the area of the following shapes. Explain the cuts that you make to transform each shape into a  $1 \times x$  rectangle. Here is a  $1 \times 1$  square for reference.



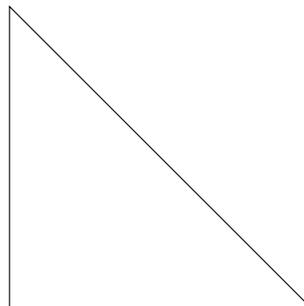
(a)



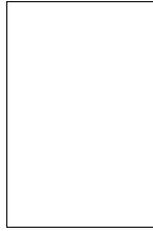
(b)



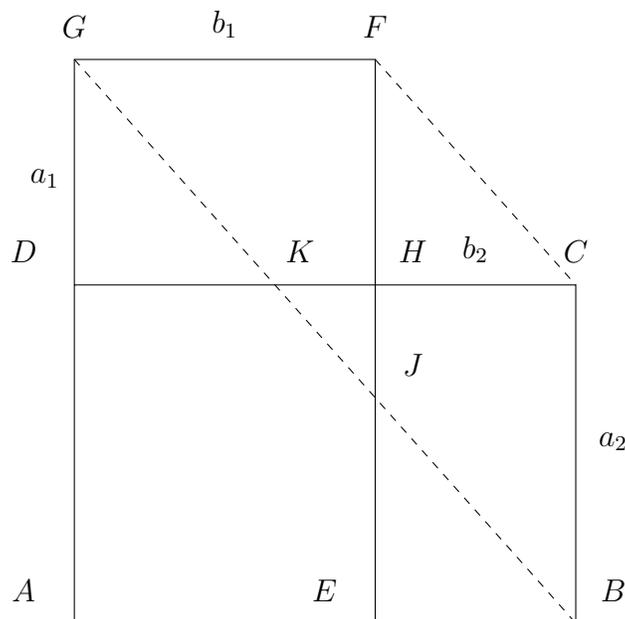
(c)



**Problem 10** (Halving the height of a Rectangle). Suppose that you have rectangle, as drawn below. Can you cut it with scissors and rearrange the pieces so that the resulting shape is a rectangle with  $1/2$  the height and 2 times the width?



**Problem 11.** Suppose that we have a rectangle of side length  $a_1, b_1$  and another rectangle of side length  $a_2, b_2$ , as shown below. Take (as a given) that if  $a_1 \times b_1 = a_2 \times b_2$ , then the lines  $GB$  and  $FC$  are parallel.



(a) Triangle  $EBJ$  is the same as what other triangle drawn on the figure?

(b) Triangle  $GJF$  is the same as what other triangle drawn on the figure?

(c) Describe the area of rectangle  $EBCH$  in terms of the triangles  $EBJ$ ,  $KCB$  and  $KHJ$ .

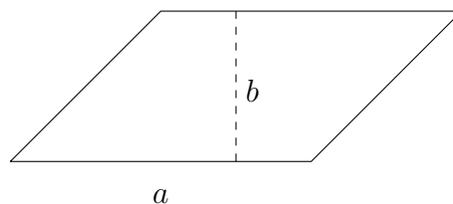
(d) Describe the area of rectangle  $GDHF$  in terms of the triangles  $GDK$ ,  $GJF$  and  $KHJ$ .

(e) How does this show that the area of  $EBCH$  is the same as the rectangle  $GDHF$ ?

Notice that this proof depended on the fact that  $GB$  and  $FC$  were parallel lines. We will get back to this issue later.

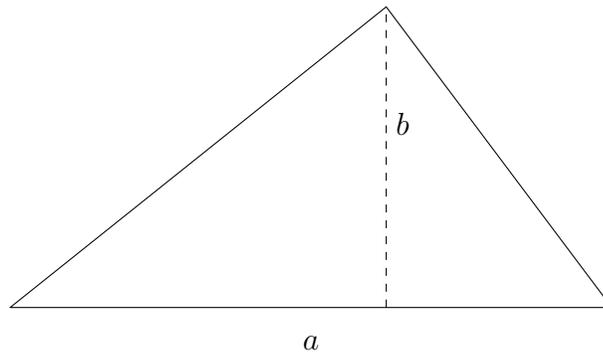
**Problem 12.** How does the above problem show that the area of a rectangle with a width of  $a$  and a height of  $b$  has the area of  $a \times b$ . (Remember the definition of area that we used before!)

**Problem 13.** A parallelogram is a shape where all four sides are parallel. Can you show using the scissor equivalence that the area of this parallelogram is the same as a rectangle of width  $a$  and side length  $b$ ?



**Problem 14.** A triangle is a shape with three sides. Can you show using the scissor equivalence principle that the area of the triangle is  $1/2$  the area of rectangle with a width of  $a$  and a side length of  $b$ . (Hint: Make a copy of the triangle in such a way that

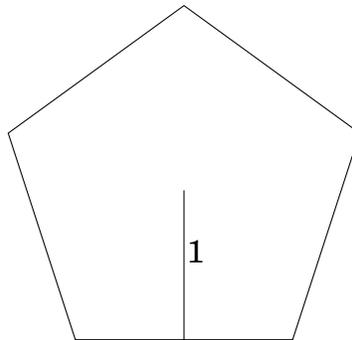
the resulting shape is a parallelogram. Then apply the previous problem)



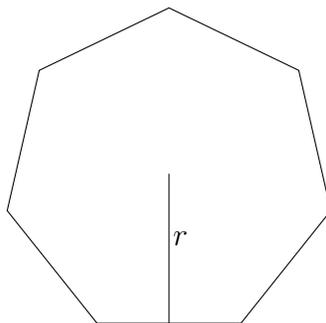
**Problem 15.** Write down (in full sentences) a method for turning any triangle into a rectangle by cutting and rearranging the parts.

A **regular polygon** is one where all of the sides have the same length, and that every angle between sides are the same. The perimeter of a shape is the sum of all 4 side lengths. The Apothem is the length of the line drawn from the center of the shape perpendicularly to the side.

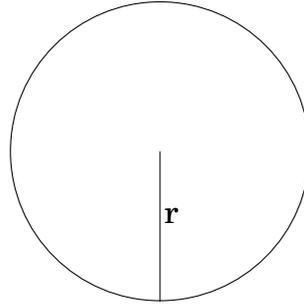
**Problem 16.** Suppose that you know that the apothem of this pentagon is 1, and the perimeter of the pentagon is  $P$ . Then what is the area of the pentagon? (Hint: Cut the pentagon into many smaller triangles, and compute the area of the triangles, then add them back up)



**Problem 17.** Suppose that you know that apothem of this regular heptagon is  $r$ , and the perimeter is  $P$ . What is the area of the shape?



**Problem 18.** Suppose that you know that the perimeter of the circle is  $C$  and the length of the radius is  $r$ . What might be a good guess for the area of the circle?



**Problem 19.** Using the fact that the perimeter of the circle (called the circumference) is  $2\pi r$ , find a formula for the area of a circle.

**Problem 20** (Proving the lines are parallel). We will show that the triangle  $FHC$  is similar to the triangle  $GAB$ , which will show that the lines  $FC$  and  $GB$  are parallel. We know the following fact:

$$a_1 \times b_1 = a_2 \times b_2$$

- (a) A shape  $A$  is similar to a shape  $B$  if you can multiply the side lengths of  $A$  by some number,  $c$ , to get the shape  $B$ . Explain why if  $\frac{FH}{HC} = \frac{GA}{AB}$ , then the triangles are similar.
- (b) Express  $FH$ ,  $HC$ ,  $GA$  and  $AB$  in terms of the lengths  $a_1$ ,  $b_1$ ,  $a_2$  and  $b_2$ .
- (c) Using some algebra, show that  $\frac{FH}{HC} = \frac{GA}{AB}$ . Conclude that the two triangles are similar, and  $FC$  is parallel to  $GB$ .