Stirlin!g's Formula

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Stirling's formula gives an approximate value for the factorial of an integer. In its simplest form, this approximation reads:

$$\text{Stirling's Approximation:} \qquad n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Many times it is okay to just substitute the right hand side for $n!$, but one should be careful about what the above formula means. The $\approx$, in this case, means that the *relative error* becomes very small when n becomes very large. It is not, however, the case that the *absolute error* becomes small. In fact, it grows without bound:

$$\text{Absolute error} := \left| n! - \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \right| \to +\infty \text{ as } n \to \infty$$

$$\text{Relative error} := \left| \frac{n! - \left(\frac{n}{e}\right)^n \sqrt{2\pi n}}{n!} \right| \to 0 \text{ as } n \to \infty.$$

An even more precise statementfootnoteFor other ways of approximating $n!$, which are even more accurate, see `http://oeis.org/wiki/User:Peter_Luschny/FactorialFunction`. is that $n!$ has an asymptotic expansion

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{52480n^3} - \frac{571}{2488320n^4} + ...\right)$$

where the kth term in the expansion is of the form $c_k n^{-k}$.

Problems For Stirlin!g's Formula

1. Compute 33^{33} by hand.

2. How many digits does 1000! have?

3. (a) Show that the set $\{\frac{1000^k}{k!} \mid k \in \mathbf{N}\}$ is a bounded set.

   (b) For which value of k does $\frac{1000^k}{k!}$ achieve its maximum value? What is this value (roughly)?

   (c) How about $\frac{123456789^k}{k!}$?

4. The volume of an $2n$-dimensional hypersphere of radius r is $\frac{\pi^n r^{2n}}{n!}$. In which (even) dimension does a ball of radius 25 have the maximum volume? Approximately what is this volume?

5. A fair coin is flipped a thousand times.

   (a) What is (roughly) the chance of getting exactly 500 heads?

   (b) What is (roughly) the chance of getting between 490 and 510 heads?

6. Elementary estimates of $n!$

   (a) How many permutations of the set $\{1, 2, \ldots, n\}$ are there?

   (b) How many functions are there from the set $\{1, 2, \ldots, n\}$ to itself?

   (c) Use your answers to these questions to get an upper bound on $n!$.

   (d) Prove that $n! \geq (n/2)^{n/2}$.

7. More advanced estimates of $n!$

   (a) Explain why $\log n! = \log 1 + \log 2 + \ldots + \log n = \sum_{k=1}^{n} \log k$.

   (b) Show that $\log n!$ is less than the area of the region in the xy-plane bounded by the x-axis, the curve $y = \log x$, and the line $x = n + 1$.

   (c) Show that $\log n!$ is more than the area of the region in the xy-plane bounded by the x-axis, the curve $y = \log x$, and the line $x = n$.

   (d) Ask someone who knows how to integrate by parts what these areas are exactly.

Sorting Complexity

In computer science, one the most common tasks is to sort a list of numbers by size. Assume the only way to order things is by comparing two elements, and nothing is known about the list in advance. One way to think about the problem is that there are $n!$ possible orderings the list could be in, and one needs to decide which one. Any procedure for deciding which ordering the list is in can be represented as a binary tree. (Why?)

1. Use the above perspective to give a lower bound on the smallest number of comparisons it could possibly take to distinguish all $n!$ orderings of a list.

2. Using the previous problem and Stirling's approximation, show that to sort a list of n elements it takes at least on the order of $n \log n$ comparisons.