WHAT IS A DISTANCE?

MATH CIRCLE (INTERMEDIATE) 04/21/2013

In mathematics, a *distance* is a certain function which tells you how far two objects are in a certain sense.

Here are some examples of distances you are probably familar with:

(1) If a and b are two points on the real number line, the distance between these two points is
$$d(a, b) =$$

(2) If (x_1, x_2) and (y_1, y_2) are two points on the coordinate plane, the distance between them is
$$d((x_1, y_1), (x_2, y_2)) =$$

(3) Let A and B be two points on the unit circle $x^2 + y^2 = 1$ such that $\angle AOB = \theta$, where O is the center of coordinate system. The distance between A and B along the circle is the length of the (shorter) arc connecting the two points. Thus
$$d(A, B) = .$$

Of course, not any rule that gives a number for a pair of objects can be considered a distance. Let's take a look at some examples...

(1) Let a and b be two numbers on the real line. Suppose that someone wants to call the value $a + b$ the distance between these two numbers.
  (a) What is the "distance" from a number a to itself, i.e., $d(a, a)$, according to this formula? Does this agree with what we expect from a distance?

  (b) What is the "distance" from -1 to -2 according to this formula? Does this agree with what we have in the examples of the distances above?

  (c) Formulate a property of distance that tells us what values the distance can assume.

(2) Consider the same scenario again: Let a and b be two numbers on the real line. Suppose that someone wants to call the value $a + b$ the distance between these two numbers.
  (a) What is the "distance" from -2 to -2 according to this formula? Do you see something "wrong" with this?

  (b) Formulate a property of distance that would resolve this issue.

(3) Let a and b again be two numbers on the real line. Suppose that someone wants to call the value $d(a, b) = a - b$ the distance between these two numbers.

(a) What is the relation between the "distances" $d(1, 3)$ and $d(3, 1)$? Does this agree with what we have in the examples of distances above?

(b) What is the relation between the "distances" $d(a, b)$ and $d(b, a)$? Does this agree with what we have in the examples of distances above?

(c) Formulate a property of distance that relates $d(a, b)$ and $d(b, a)$ for any a and b.

(4) To formulate the third property of distance, let's consider the example as before, where for two numbers a and b on the real line, the "distance" between a and b is $d(a, b) = a + b$.

(a) Consider the points on the real line 1, 3, and 5. What are the "distances" between these points? (You should have three answers.) Do you see a potential problem?

(b) Consider the points on the coordinate plane $(1, 1)$, $(4, 1)$, and $(1, 5)$. What are the "distances" between these points? Do you see a potential problem?

(c) Consider three arbitrary points on the coordinate plane: A, B, and C. What does the triangle inequality tell us about the distances between these three points?

(d) Let $a, b,$ and c be three numbers on the real line. Compare $d(a, c)$ with $d(a, b) + d(b, c)$. Does this agree with what we know about distance?

(e) Formulate a property of distance that relates $d(a, c)$ with $d(a, b) + d(b, c)$.

(5) **You are now ready to give a definition of a distance.**
Mathematicians call $d(A, B)$ a *distance* (or distance function) if the following properties are satisfied:

(a) (see Problem 1)

(b) (see Problem 2)

(c) (see Problem 3)

(d) (see Problem 4)

(6) For each of the following examples, decide whether $d(a, b)$ is a distance. If yes, show that all four properties in the definition of a distance are satisfied. If not, specifically explain which property (or properties) fail.

(a) Let a and b be two numbers on the real line and

$$d(a, b) = \begin{cases} 1, & \text{if } a \neq b \\ 0, & \text{if } a = b \end{cases}.$$

(i) (Property 5a)

(ii) (Property 5b)

(iii) (Property 5c)

(iv) (Property 5d)

(b) Let (x_1, y_1) and (x_2, y_2) be two points on the plane and

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - x_2|.$$

(i) (Property 5a)

(ii) (Property 5b)

(iii) (Property 5c)

(iv) (Property 5d)

(c) Let $a+b$ be two numbers on the real line and $d(a,b) = 1 + |a-b|$.

(i) (Property 5a)

(ii) (Property 5b)

(iii) (Property 5c)

(iv) (Property 5d)

(d) Let a and b be two numbers on the real line. Define

$$d(a,b) = \begin{cases} 0, & \text{if } a = b \\ 1, & \text{if } |a-b| \leq 1 \\ 2, & \text{if } |a-b| > 1 \end{cases}.$$

(i) (Property 5a)

(ii) (Property 5b)

(iii) (Property 5c)

(iv) (Property 5d)

(7) Interestingly, one can define distances between other types of objects. First, distance between words: positions that do not coincide.
Let's define a distance between two English words to be the number of places in which the words have different letters when the words are left aligned. Here are some examples:

$$d(\text{STAR}, \text{START}) = 1$$
$$d(\text{STAR}, \text{JAR}) = 4$$

Find the following distances:
(a) d(JAR, BAR)=

(b) d(BELL,CELL) =

(c) d(SUN,STAR) =

(8) Show that the distance between the words defined above satisfies the axioms of distance.
(a) (Property 5a)

(b) (Property 5b)

(c) (Property 5c)

(d) (Property 5d)

(9) Find three words which have distance 1 to the word FLOW.

(10) Find three words which have distance 2 to the word SLOW.

(11) Find as many words as you can that have the same (nonzero) distances to the word STUNT and to the word STAKE .

(12) Lewis Carroll invented a game where one English word has to be turned into another via a series of changes. Each change should produce a new word which is different from the previous one in only one position. Here is an example of a valid transformation from MILK into ROLL:
MILK
MILL
DILL
DOLL
ROLL

Starting with the word FLOW, see what you can turn it into in 4 steps.

(13) Here's another example of distance between words: are there any letters in common? Define the distance between two English words as follows:
   (a) distance is 0 if two words coincide:
   (b) distance is 1 if two words have at least one letter in common;
   (c) distance is 2 if the two words have no letters in common.

   Check that all the axioms of distance are satisfied for the distance defined this way.
   (i) (Property 5a)
   (ii) (Property 5b)
   (iii) (Property 5c)
   (iv) (Property 5d)

(14) Find 3 words which have distance 1 to the word DISTANCE.

(15) Find 3 words which have distance 2 to the word DISTANCE.

(16) Find 3 words that have the same distance to CAT and to DOG. (Indicate the distance in each case).

(17) Distance between vertices of a graph:
For a (connected) graph, define the distance between two vertices as the smallest number of edges along which one can travel from the first vertex to the second one.

Check that the axioms of distance are satisfied.

(a) (Property 5a)

(b) (Property 5b)

(c) (Property 5c)

(d) (Property 5d)

(18) Let X be the set of all people in the world. (We might have to exclude some true hermits.) Define the distance between two people A and B to be the smallest number n such that there is a chain of n people $P_1, P_2, P_3, ..., P_n$, where P_1 knows P_2 , P_2 knows P_3 , etc., and the first person P_1 is A and the last person P_n is B. The distance between a person and his or herself is defined to be 0. This function satisfies the four properties of distance. (To make symmetry work, we will assume that if P knows Q, then Q knows P.)

(a) If you wanted to find the distance between people on Earth by using a graph, what would the vertices be? The edges? How many of each would there be?

(b) What is the maximal distance between any two people on Earth? Make reasonable assumptions.
(Sociologists have performed experiments where they ask a person, say, in California, to mail a letter to a total stranger, say, in Maine. The instructions are to mail the letter to someone closer, who can in turn, forward the letter along its way. Apparently most people are within 4-6 contacts of each other.)

(19) Are these distance functions? Justify your answer.
(a) $d(a, b) = |a^2 - b^2|$

(b) For two points $a = (x_1, y_1),\ b = (x_2, y_2)$, $d(a, b) = \max(|x_1 - x_2|, |y_1 - y_2|)$

(c) For two points $a = (x_1, y_1),\ b = (x_2, y_2)$, $d(a, b) = \min(|x_1 - x_2|, |y_1 - y_2|)$

- Review Problems -

(20) Prove that a graph in which any two vertices are connected by one and only one simple path is a tree.

(21) Does a graph with 5 vertices of degrees 4, 4, 4, 4, and 2 exist? If so, draw it. If not, prove it.

(22) There are 41 towns in a country. Each of them is connected to exactly half the others by a single road. What is the maximum number of roads that can be closed in such a way that one can still reach each town from any other?