Rational, Algebraic & Transcendental Numbers

UCLA Math Circle

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Definition 1. *A real number x is a **rational number** if it can be written as $\frac{a}{b}$ with integers a, b.*

Definition 2. *A complex number z is an **algebraic number** if there exists integers $a_0, a_1, \dots, a_n$ such that*

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Definition 3. *A complex number z is called **transcendental** if it is not algebraic.*

We use the following notations for the set of real, rational and algebraic numbers.

$$\begin{aligned} \mathbb{R} &:= \text{ The set of real numbers.} \\ \mathbb{Q} &:= \text{ The set of rational numbers.} \\ \overline{\mathbb{Q}} &:= \text{ The set of algebraic numbers.} \end{aligned}$$

Definition 4. *A continued fraction is a fraction of the form*

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdot}}},$$

where a_0 is an integer and a_k is a positive integer for all $k \geq 1$. We use the notation $[a_0; a_1, a_2, a_3, \dots]$ to represent it, and use $[a_0; , a_1, \dots, a_k]$ to represent the partial fraction

$$a_0 + \cfrac{1}{a_1 + \frac{1}{\frac{\cdots}{a_{k-1} + \frac{1}{a_k}}}}.$$

Proposition 1. *Let $x = [a_0; a_1, a_2, \dots]$ be the continued expansion of x and define two sequences P_n, Q_n by*

$$P_0 = a_0, P_n = a_n P_{n-1} + P_{n-2},$$
$$Q_0 = 1, Q_n = a_n Q_{n-1} + Q_{n-2}.$$

Then $\frac{P_n}{Q_n}$ is the n^{th} partial sum $\frac{P_n}{Q_n} = [a_0, a_1, a_2, \dots, a_n]$. Furthermore, we have the estimate

$$\left| x - \tfrac{P_n}{Q_n} \right| \leq \tfrac{1}{Q_n Q_{n+1}}.$$

Theorem 1 (Euler, Lagrange)**.** *A continued fraction $[a_0; a_1, a_2, \dots]$ represents the solution of a quadratic equation if and only if it is periodic, i.e. there exists n such that $a_k = a_{k+n}$ for all $k \geq 1$.*

Definition 5. *A **Liouville number** is a real number x such that for every positive integer n, there exist integers p and q with $q > 1$ such that*

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

Theorem 2. *All Liouville numbers are transcendental.*

Exercises

1. Let $u, v \in \mathbb{Q}$. Show that $u \pm v, uv, u/v$ are all in $\mathbb{Q}$. Give an example of $u, v \in \mathbb{Q}$ such that $u^v \notin \mathbb{Q}$.

2. Write the following decimals as fractions (the vinculum sign means the digits are repeating)

   - 0.333
   - $0.\overline{6}$
   - $0.\overline{285714}$
   - $0.\overline{9876543210}$

3. Show that a decimal $0.a_1a_2a_3\dots$ is a rational number if and only if the digits are repeating, i.e. there exists $n \geq 1$ such that $a_k = a_{k+n}$ for all $k \geq 1$.

4. Write out the continued fraction representation of the following decimals

   - 0.333
   - $0.\overline{6}$
   - $0.\overline{285714}$
   - $0.\overline{9876543210}$

   What can you say about the continued fraction of a rational number?

5. Show that $\sqrt{3} \in \overline{\mathbb{Q}}$ is not a rational number and write out its continued fraction expansion.

6. Write the following continued fractions as algebraic numbers

   - $[1; 1, 1, \overline{1}]$
   - $[5; \overline{1, 1, 3, 5, 3, 1, 1, 10}]$
   - $[0; \overline{1, 4, 1, 1}]$

7. Write out the first 5 digits of the continued fraction expansion of $\sqrt[3]{2}$.

8. Let $u, v \in \overline{\mathbb{Q}}$. Show that $u \pm v, uv, u/v$ are all in $\overline{\mathbb{Q}}$. Give an example of $u, v \in \overline{\mathbb{Q}}$ such that $u^v \in \overline{\mathbb{Q}}$. What about an example such that $u^v \notin \overline{\mathbb{Q}}$.

9. Write the complex number z as $x + iy$ with real numbers x, y. Show that z is algebraic if and only if x and y are both algebraic.

10. Show that the following decimal, which is called *Liouville's constant*, is a Liouville number, hence transcendental

$$\sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000\ldots$$

11. Use continued fractions to produce more examples of Liouville numbers.

12. Prove Theorem 2 following the steps below

    (a) Show that Liouville numbers are not rational.

(b) Let $[a,b]$ be a fixed interval $f(x)$ a polynomial. Show that there exists a constant $C_{f,a,b}$ depending on a,b,f such that for any $c,d \in [a,b]$

$$\left|\frac{f(c)-f(d)}{c-d}\right| < C_{f,a,b}.$$

(*Hint: induction on the degree of* f).

(c) Let α be an algebraic number and $f(x)$ a polynomial of degree n with integral coefficients such that $f(\alpha) = 0$ and $f(x)$ has no rational root. Suppose $M = C_{f,\alpha-1,\alpha+1}$ is the constant as in the previous exercise and define

$$A = \min\{1, 1/M\}.$$

Show that for any $\frac{p}{q} \in \mathbb{Q}$, we have

$$\left|\alpha - \frac{p}{q}\right| > \frac{A}{q^n}.$$

(*Hint: if* $p/q \in [\alpha-1, \alpha+1]$, *consider* $|\frac{f(\alpha)-f(p/q)}{\alpha-p/q}| < M$.

(d) Show that Liouville numbers are not algebraic numbers.

13. (*Challenge*) Prove that

$$\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = [1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, \ldots, 1, 2n, 1, \ldots].$$

(*Hint: see* `http://arxiv.org/pdf/math/0601660.pdf` *for a detailed proof using recursion and simple calculus.*)