Trees, Pigeonholes, and Dandelions

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ORMC Advanced 3

1 Trees

"Begin at the beginning," the King said gravely, "and go on till you come to the end: then stop."

Lewis Carroll, *Alice in Wonderland*

In this section, you will use your familiarity with bitstreams to study trees. Take it slow, enjoy the scenery, and, oh, don't forget to look for the dandelions: some of the trees you invent may look more like dandelions than oaks.

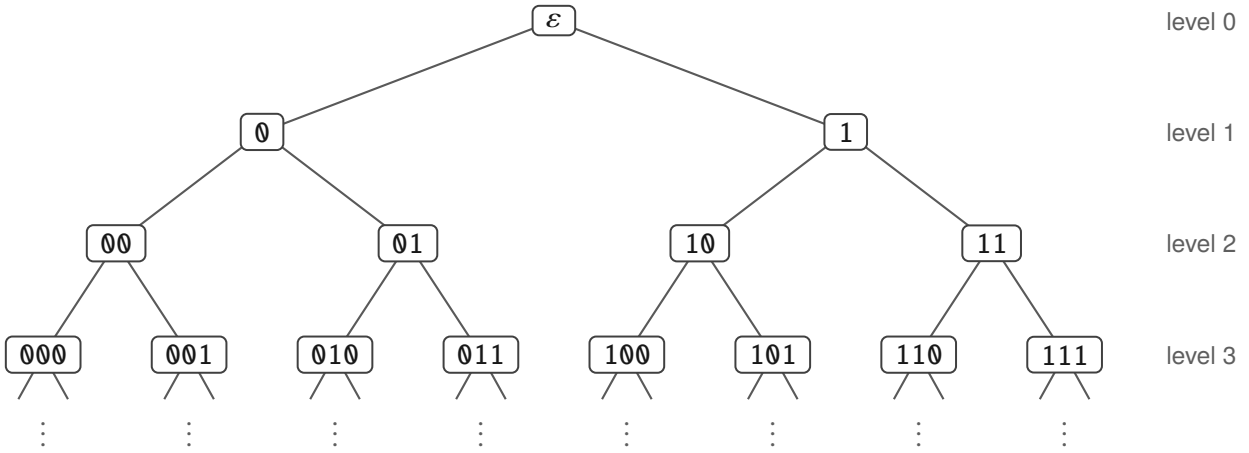


Figure 1. Levels 0 through 3 of the Cantor tree.
The root ε represents the empty string.

Exercise 1.1. Consider the Cantor tree (Figure 1). Each node is associated with a finite binary string. For example, the string 01011 corresponds to the node we reach by starting at the root, and then walking down 0, then 1, then 0, then 1, then 1. What do infinite branches starting at the root in this Cantor tree correspond to?

Exercise 1.2. Draw five examples of finite trees, each growing downward from its root. Try to draw ones that look different from each other! Write the size, or *width*, of each level (level 0 consists of the root), and write the *height* of each tree: the largest N for which level N is not empty.

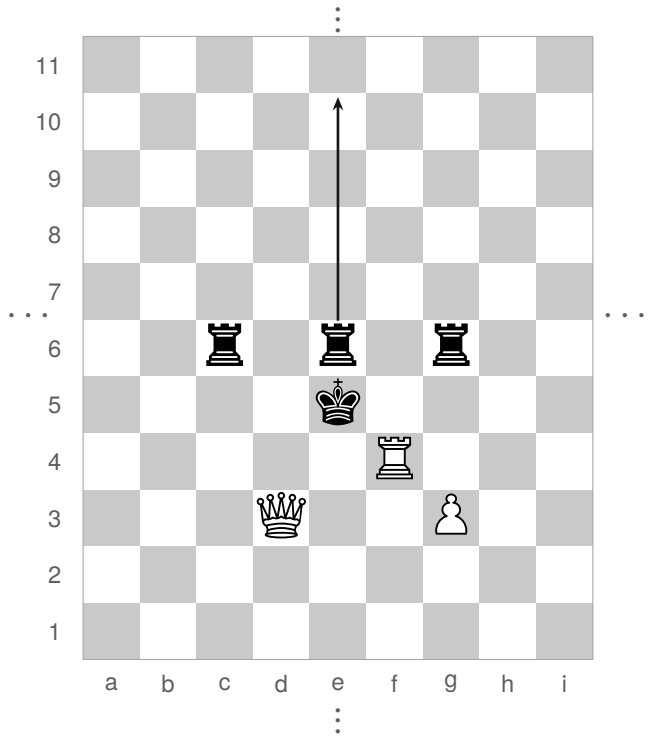
Exercise 1.3. Describe an example of a rooted tree that has infinitely many nodes but no infinite branch. Draw a representative finite portion of your tree. Here, an infinite branch starts at the root and repeatedly moves downward to a child, continuing forever.

Exercise 1.4 (König's lemma). Suppose we have an infinite tree T growing downward from a root; *infinite* means that it has infinitely many vertices. Suppose every node has only finitely many children. Such a tree is called *finitely branching*. For a rooted tree, this is equivalent to saying that every level has finite width, although the widths need not have a common finite bound. Draw an example of such a tree. Prove that it **must** have an infinite branch. Contrast this with the previous exercise.

Exercise 1.5. Suppose we have a tree T growing downward from a root. Do **not** assume that it has finite-width levels. Suppose T has **no** infinite branches. Describe an example of such a tree that has infinite height, meaning that every level $n = 0, 1, 2, \ldots$ is nonempty. Draw a representative finite portion.

Exercise 1.6. Let's think about the tree T in the previous problem in a different way. Suppose you are standing at the root. At every leaf—a node with no children—there is a bomb. Each second, you must move downward to a child: no waiting and no backtracking. Reaching a leaf triggers its bomb.

Explain why you must eventually reach a bomb, yet for every positive integer N you can choose a path that keeps you alive for at least N seconds. How can both statements be true? Relate this mind-boggling idea to the infinite chess position in Figure 2 on the next page.



▼ **BLACK TO MOVE**

The pictured rectangle is only a window: the chessboard continues infinitely in all four directions. All unpictured squares are empty.

Pieces move as usual; White's pawn moves upward. There is no pawn promotion, and we ignore the fifty-move and repetition draw rules.

This position has no White king, which is allowed here. White's goal is to checkmate Black's king.

Think about Exercise 1.6. With a suitable strategy, White can force eventual checkmate, but Black can make the required number of moves arbitrarily large. How is that possible?

Figure 2. How long can Black postpone checkmate?

Position adapted from C. D. A. Evans and J. D. Hamkins, *Transfinite game values in infinite chess*, Figure 3 (arXiv:1302.4377).

Exercise 1.7. Suppose we have a tree T growing downward from a root, with finite-width levels. Suppose T has **no** infinite branches. Prove that T has finite height: there is an integer N such that every node is at a level less than N. *Hint:* König's lemma.

Exercise 1.8. Consider the Cantor tree, the infinite binary tree. A runner, Rolfson, is standing at its root. Unfortunately, someone evil has placed pressure mines at some of its nodes. Each move must go downward to a child.

Suppose that for every natural number $n \geq 1$, there is a mine-free path p_n from the root to level n. For example, Rolfson can find a safe path p_{100} to level 100 and a safe path p_{1000} to level 1000, although the second path need not extend the first.

Prove that there is an infinite mine-free branch, so Rolfson can avoid all the mines and run on forever into the sunset. Contrast this with the previous exercises. Try to make sense of it all in your head.

(This is a rephrasing of the extremely important fact that Cantor space is a compact topological space. We will cover these concepts in more detail later this quarter.)

Exercise 1.9. Return to the Cantor tree, with Rolfson at its mine-free root. Do **not** assume that safe paths reach every finite level, as in Exercise 1.8. Instead, suppose that there is at most one mine at each level $n \geq 1$. Prove that an infinite mine-free branch exists, so Rolfson can run forever if he always chooses the correct left or right turn.

2 Pigeonholes

Food is fuel. Now shut up and eat your garbage.
Django, *Ratatouille*

In this section, you will encounter a new stream, the *ratstream*. And no, unlike the bitstream (which is a stream of bits), this is not a stream of rats. But first, you've got to think about how to put pigeons in their respective pigeonholes.

Exercise 2.1. Suppose I have infinitely many balls, labeled $1, 2, 3, \ldots$, and two bins. Prove that no matter how I place all the balls in the bins, at least one bin contains infinitely many balls. This is a very important fact called the *infinitary pigeonhole principle*.

Exercise 2.2. If you didn't like the non-physicality of infinitely many balls, what is the minimum number of balls so that, no matter how I place all of them in two bins, at least one bin contains **at least three balls**?

Exercise 2.3. Suppose I have six dots, arranged as the vertices of a regular hexagon. I draw one line segment joining each pair of distinct vertices. How many segments do I draw? Suppose I color each segment blue or red. Show that, no matter how I color them, three of the original six vertices form a triangle whose three sides have the same color. Intersections of crossing segments do not count as new vertices.

Exercise 2.4. Suppose I have infinitely many balls, labeled $1, 2, 3, \ldots$, with one spring joining every pair of distinct balls. Each spring is colored blue or red. Prove that I can choose infinitely many balls and place them into a bin so that all springs with both endpoints in that bin have the same color.

Exercise 2.5. What if, in the previous problem, I had three colors: blue, red, and yellow? What about four colors?

Exercise 2.6 (Challenge †). Suppose I have infinitely many balls in a line, labeled $1, 2, 3, \ldots$, with each ball colored blue, red, or yellow. Note that here the *balls* are colored, whereas above the *springs* between pairs of balls were colored. Prove that there are three balls of the same color, with labels $x < y < z$, such that

$$x + y = z.$$

Such triples are called *Schur triples*. *Hint:* Use the coloring of the balls to get a coloring of the springs. Make sure your construction gives two distinct summands, so that $x < y$.

Exercise 2.7. You may use the previous exercise as a black box. Prove that there is an integer $N \geq 1$ such that every coloring of $\{1, 2, \ldots, N\}$ with blue, red, and yellow contains a Schur triple: three integers $x < y < z \leq N$ of the same color with $x + y = z$. *Hint:* Use König's lemma.

Exercise 2.8. A *ratstream* is an infinite sequence of rational numbers. For example,

$$(a_1, a_2, a_3, a_4, \ldots) = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\right).$$

Here's one way to visualize ratstreams as buildings. Draw a horizontal number line. For $n = 1, 2, 3, \ldots$, use the interval

$$\left[-\frac{1}{n}, -\frac{1}{n+1}\right]$$

as a rectangle's base, and extend the rectangle vertically from coordinate 0 to coordinate a_n. If $a_n < 0$, it extends *below* the horizontal axis; if $a_n > 0$, it extends above it. Its actual height is $|a_n|$. If $a_n = 0$, the rectangle reduces to its base segment.

Draw the first several rectangles for the ratstream above and indicate how the pattern continues. Then do the same for a new ratstream of your own.

Exercise 2.9. A ratstream $(a_1, a_2, \ldots)$ is *weakly increasing* if $a_1 \leq a_2 \leq \cdots$. Define *weakly decreasing*. Then define an *infinite weakly increasing subsequence* of a ratstream. Your definition should explain how the selected terms retain their original order.

Exercise 2.10. Prove that every ratstream has either an **infinite** weakly increasing subsequence or an **infinite** weakly decreasing subsequence. It can have both, but we just want to guarantee that it has at least one.

(This is one of the most important theorems in undergraduate real analysis, the "monotone subsequence theorem".)

For Exercises 2.11–2.13: The lists have distinct entries, subsequences are finite, and *increasing* and *decreasing* mean *strictly* increasing and strictly decreasing.

Exercise 2.11. Consider the sequence of ten natural numbers

$$4,\ 6,\ 1,\ 9,\ 2,\ 3,\ 8,\ 5,\ 10,\ 7.$$

Find a longest increasing subsequence (LIS) that ends at 8, and a longest decreasing subsequence (LDS) that ends at 5. Go from left to right, calculating the lengths u_k and d_k of an LIS and an LDS ending at each position k. Why is it helpful, and computationally efficient, to work from left to right?

(This is a prototypical example of something called a dynamic programming algorithm.)

k	1	2	3	4	5	6	7	8	9	10
a_k	4	6	1	9	2	3	8	5	10	7
u_k										
d_k										

Exercise 2.12. What do you notice about the pairs (u_k, d_k) for $k = 1, \ldots, 10$? Use the insights from your computations to prove that every list of ten distinct integers contains an increasing subsequence of length **four** or a decreasing subsequence of length **four**.

Exercise 2.13. Let $n \geq 2$ be an integer. What is the **smallest** length of a list of **distinct integers** that guarantees a strictly increasing subsequence of length n or a strictly decreasing subsequence of length n? Prove that your length works and that no smaller length always works.

Exercise 2.14. Above you saw infinitary and finitary versions of the same theorems. Which did you like more?