LAMC Junior Circle **April 28, 2013**

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Proofs

What is a PROOF?

A proof is a list of steps that explains how you know that something is true.

Well, then what is a STEP?

Good question!

Every step has two parts: the claim and the reason. The reason explains how you know that the claim is true.

EXAMPLE

Claim:	My dad's brother is my uncle.	Reason:	definition of "uncle"

There exist three kinds of STEPS:

1. The "Given" Step: Givens are the things you want the reader to assume are true. These go in the first step of your proof and you only need to say "Given" as your reason.

2. The “Body” Steps: These steps are things you can show are true based on the givens. It should be very easy for the reader to understand why each body step is true.

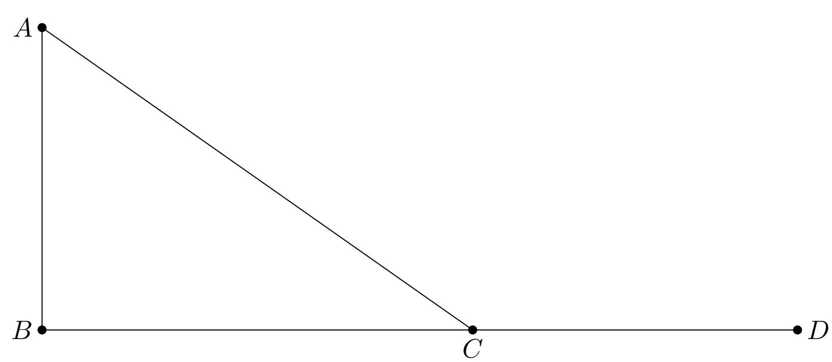
3. The “Conclusion” Step: This is the final step of your proof. The claim part of this step is the thing you are proving is true.

That’s a lot of new words. Can I see an example?

OK!

EXAMPLE

Given that the triangle ABC is a right triangle, that the point C belongs to the straight line BD, and that $\angle BAC = 55°$, prove that $\angle ACD = 145°$.



Claim	**Reason**
$\triangle ABC$ is a right triangle, $\angle BAC = 55^\circ$	Given.
$\angle ABC = 90^\circ$	Definition of a right triangle.
$\angle ABC + \angle ACB + \angle BAC = 180^\circ$	The sum of the angles of any triangle in the Euclidean plane equals 180°.
$\angle ACB = 35^\circ$	$35^\circ + 55^\circ + 90^\circ = 180^\circ$
$\angle ACB + \angle ACD = 180^\circ$	$\angle ACB$ and $\angle ACD$ add up to a straight angle. A straight angle measures 180° by definition.
$\angle ACD = 145^\circ$	$35^\circ + 145^\circ = 180^\circ$

Q.E.D. - *quod erat demonstrandum*, meaning "which had to be demonstrated" in Latin. Another way to mark the end of a proof is W^3, which stands for "What We Wanted". In mathematical research papers, they usually mark the end of a proof with a box $\square$.

Now it's your turn.

Definition 1 *A rectangle in the Euclidean plane is a quadrilateral such that all of its four angles are equal to one another.*

Problem 1 *Prove that each angle of a rectangle measures* 90°. *Remember to draw a Claim/Reason chart like the one in the above example.*

Now, please answer the following questions in your own words.

What is a proof?

What are the different parts of a proof?

How should you format a proof?

Why do mathematicians need to use proofs?

Recall that the *perimeter* of a polygon is the sum of its sides' lengths. We will also need the following fact to solve the problem below.

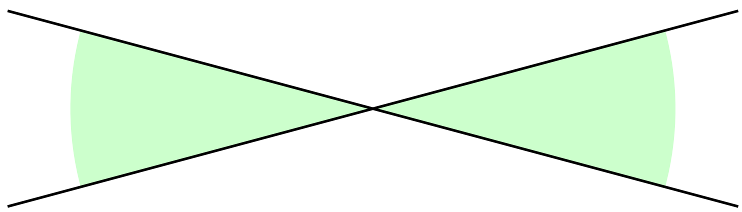
Fact 1 — The triangle inequality

The length of a side of a triangle in the Euclidean plane is always less than the sum of the lengths of the remaining two sides.

Problem 2 *Draw a picture, then use the Claim/Reason chart to prove that the length of any median of a triangle in the Euclidean plane is always less than a half of its perimeter.*

Recall the following.

Definition 2 *Two angles are called opposite, or vertical, if their sides form two straight lines as on the picture below.*



Problem 3 *Use the Claim/Reason chart to prove that opposite angles have equal size.*

Recall the following.

Fact 2 *Two triangles in the Euclidean plane are congruent if they have a pair of congruent angles, and the lengths of the sides adjacent to the angles are pairwise equal.*

$$\alpha \cong \alpha', \quad |b| = |b'|, \quad |c| = |c'|$$

Problem 4 *Use the Claim/Reason chart to prove that for any triangle in the Euclidean plane the length of a median is less than a half of the sum of the lengths of the sides it lies in between.*

Problem 5 *Use the Claim/Reason chart to prove that for any triangle in the Euclidean plane the sum of the lengths of its medians is less than the perimeter of the triangle, but greater than a half of the perimeter.*

A *quadratic equation* is an equation of the form

$$ax^2 + bx + c = 0, \text{ where } a \neq 0. \tag{1}$$

The following algorithm solves 1 (over the real numbers).

Step 1. Find the *discriminant* of the equation.

$$D = b^2 - 4ac \tag{2}$$

Step 2. If $D < 0$, then the equation has no solutions (over the real numbers).

Step 3. If $D = 0$, then the equation has one solution,

$$x = -\frac{b}{2a}.$$

Step 4. If $D > 0$, then the equation has two different solution,

$$x_1 = \frac{-b + \sqrt{D}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{D}}{2a}$$

The latter two formulae are often written as one formula,

$$x_{12} = \frac{-b \pm \sqrt{D}}{2a} \tag{3}$$

Problem 6 *Solve the following equations. Check the solutions by plugging them back into the equations.*

- $x^2 + x - 6 = 0$

- $5x^2 + 10x + 5 = 0$

- $x^2 + 4x + 5 = 0$

- $x^2 - 11x + 30 = 0$

- $x^2 - 2.5x + 1 = 0$

Problem 7 *Find the numerical value of the following infinite expression.*

$$\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\dots}}}}$$

Problem 8 *Find the numerical value of the following infinite expression (called a continued fraction).*

$$2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\dots}}}}$$

Problem 9 *Solve the following equation.*

$$x^2 - 2 = 0$$

Hint: the algorithm on page 10 is not the best way to solve this one.

Problem 10 *Use the Claim/Reason chart to prove that*

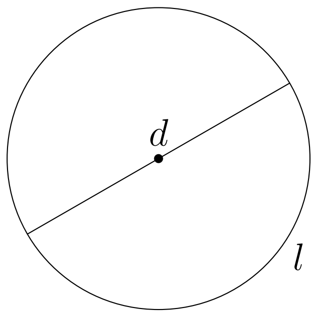
$$\sqrt{2} > 1.$$

Problem 11 *Prove that $\sqrt{2}$ is not a rational number (cannot be represented as a ratio p/q of two integers having no common factors).*

Problem 12 *Find the numerical value of the following continued fraction.*

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}$$

One of the most important numbers in mathematics and physics is $\pi = 3.14159265358979323846\ldots$ For example, for any circle in the Euclidean plane, the ratio of its circumference, l, and its diameter, d, equals π.



$$\frac{l}{d} = \pi$$

The number π is not rational. (It is not even algebraic, whatever this means.) However, it is easy to compute using continuous fractions.

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \ldots}}}}} \tag{4}$$

Indeed,

$$\cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + 4^2}}}} = 3.156...$$

which is already quite a good approximation.

Homework Problem 1 *Compute a few more consecutive approximations of π given by the formula 4. Use a calculator only at the very last step of each computation. You will lose a lot of precision otherwise!*

Homework Problem 2 *Think of another way to find good approximations of π.*