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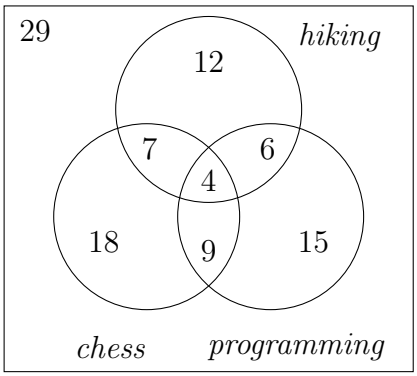
Introduction to Set Theory

Warm-up

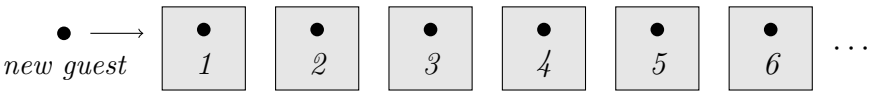
Problem 1 *One hundred students of a math circle were asked which of three activities they enjoy: chess, programming, and hiking. The diagram below shows the resulting counts. For example,* 18 *students enjoy chess but neither of the other two, and* 29 *enjoy none of the three.*



How many of the 100 *students*

- *enjoy chess?* =
- *enjoy both hiking and programming?* =
- *enjoy programming but not chess?* =
- *enjoy exactly two of the three activities?* =
- *enjoy at least one of the three?* =

Problem 2 *The Grand Hilbert Hotel has infinitely many rooms, numbered* $1, 2, 3, \ldots$ *without end. Tonight every room is occupied. A late traveler walks in and asks for a room. "Not a problem," says the manager. Nobody will share a room, and nobody will be left without one. Can the manager be right? Write down your gut answer and one sentence of reasoning. We will come back to the hotel later in the packet.*



Your guess (yes/no):_____ **because** _____

_____

What is a set?

A **set** is a collection of distinct objects, called its **elements**. Note that this is not much of a definition: all we did was trade the word *set* for the word *collection*. The notion of a set is too fundamental to be explained in simpler terms, so we explain it by example instead.

The set F of all factors of the number 6 has four elements:

$$F = \{6, 3, 2, 1\}.$$

The curly braces mean that the only thing that matters is which objects belong to the set: neither order nor repetition changes it. Thus $\{1, 2, 3, 6\}$ and $\{6, 3, 3, 2, 1\}$ are the same set F. The fact that 3 is an element of F is written $3 \in F$; since 5 is not an element, we write $5 \notin F$. When order does matter, we use parentheses and speak of a *list*: $(1, 2)$ and $(2, 1)$ are different lists, but $\{1, 2\}$ and $\{2, 1\}$ are the same set.

Problem 3 *True or false?*

- $2 \in \{1, 2, 3\}$
- $\{2\} \in \{1, 2, 3\}$
- $\{1, 1, 2, 3\} = \{1, 2, 3\}$
- $\{1, 2\} \in \big\{\{1, 2\},\ 3\big\}$

Some sets are so important that they have permanent names.

- $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of **natural numbers**. (Some books let 0 be a natural number and some do not; it is a matter of convention. Today, $0 \notin \mathbb{N}$.)
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of **integers**.
- $\mathbb{Q}$ is the set of **rational numbers**: all fractions $\frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$.
- $\mathbb{R}$ is the set of **real numbers**: all points of the number line.
- A set with no elements at all is called the **empty set** and is denoted $\varnothing$ or $\{\}$. Note that we say *the* empty set and not *an* empty set, since there is only one.

A set can also be described by a rule. In math language, we write $\{x :$ condition on $x\}$, read "the set of all x such that the condition holds." For example, $\{n \in \mathbb{N} : n \text{ is odd}\} = \{1, 3, 5, 7, \dots\}$, and the same set can be written more compactly as $\{2k - 1 : k \in \mathbb{N}\}$. Formal sentences often use two more symbols: $\forall$ ("for all") and $\exists$ ("there exists").

Problem 4 *Translate each set into the other notation.*

- $\{x \in \mathbb{Z} : x^2 \leq 9\} =$
- *the set of all even natural numbers* $=$
- $\{10, 100, 1000, 10000, \dots\} =$

If every element of a set A also belongs to a set B, we say that A is a **subset** of B and write $A \subseteq B$. If moreover $A \neq B$, we call A a **proper subset** of B and write $A \subset B$. For example, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. Be careful to distinguish $\in$ from $\subseteq$: for the set $\{1, 2, 3\}$ we have $2 \in \{1, 2, 3\}$ and $\{2\} \subseteq \{1, 2, 3\}$, but not the other way around.

Problem 5 *Is there a set that is a subset of* every *set? Is there a set that is a* proper *subset of every set?*

The number of elements of a finite set is called its **cardinality**. In math language, the cardinality of a set A is written $|A|$. For example, $|\{C, A, T\}| = 3$.

Problem 6 *Find each cardinality.*

- $|\{2, 0, 2, 6\}| =$
- $|\varnothing| =$
- $|\{\varnothing\}| =$
- $\left|\left\{\varnothing,\ a,\ b,\ \{a\},\ \{a, b\},\ \{a, a\},\ \{b, a, b\}\right\}\right| =$

Every set has subsets, and all of them can be gathered into a single new set. The **power set** of A, written $\mathcal{P}(A)$, is the set of all subsets of A. For example,

$$\mathcal{P}(\{1,2\}) = \{\, \varnothing,\ \{1\},\ \{2\},\ \{1,2\} \,\}.$$

Note that $|\{1,2\}| = 2$, while $|\mathcal{P}(\{1,2\})| = 4$.

Problem 7

a. Write down all the subsets of $\{1,2,3\}$*. How many are there? (Do not forget* $\varnothing$ *and the set itself.)*

b. Fix the order $1,2,3$*. Any subset of* $\{1,2,3\}$ *can be recorded as a string of three bits: the first bit is* 1 *if the subset contains* 1 *and* 0 *if it does not, and similarly for the second and third. For example,* $\{1,3\} \leftrightarrow 101$ *and* $\varnothing \leftrightarrow 000$*. Which subset corresponds to the string* 110*? =*

c. How many different strings of n *bits are there? Use b. to find how many subsets a set of* n *elements has. =*

d. $\big|\mathcal{P}\big(\mathcal{P}(\{1,2\})\big)\big| =$

Building new sets from old

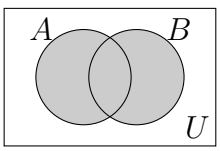
Two sets can be combined in several natural ways. The **intersection** of the sets A and B is defined as

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

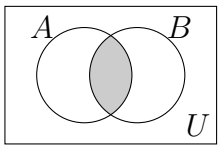
Here the symbol := means "is defined as". Translating back to English: the intersection of A and B is the set of all elements that belong to both. In the same spirit we define the **union** and the **difference**,

$$A \cup B := \{x : x \in A \text{ or } x \in B\}, \qquad A \setminus B := \{x : x \in A \text{ and } x \notin B\},$$

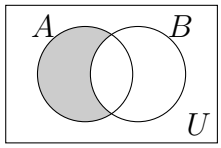
where, as always in mathematics, "or" includes the possibility of both. Finally, when all sets under discussion are subsets of a fixed **universal set** U, the **complement** of A is everything outside of it: $A^c := \{x \in U : x \notin A\}$. The pictures below, called **Venn diagrams**, show all four operations as shaded regions.



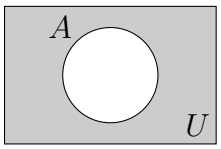
$A \cup B$



$A \cap B$



$A \setminus B$



A^c

Two sets are called **disjoint** if they have no elements in common, that is, if $A \cap B = \varnothing$.

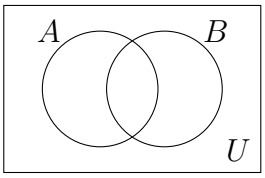
Problem 8 *Let the universal set be $U = \{1, 2, \ldots, 10\}$, let A be the set of even numbers in U, and let B be the set of multiples of three in U. Find:*

- $A \cap B =$
- $A \cup B =$
- $A \setminus B =$
- $B^c =$
- $A \cap \varnothing =$

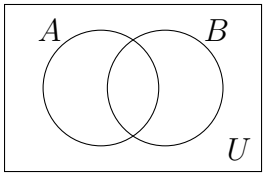
Problem 9 *Now you write the math.*

- *Define the power set in our formal notation:* $\mathcal{P}(A) :=$
- *Express "A and B are disjoint" as an equation:*
- *Write the set of all multiples of five in set-builder notation:*

Problem 10 *Shade each of the four regions below. What do you notice?*
a. Shade $(A \cup B)^c$ *on the first diagram and* $A^c \cap B^c$ *on the second.*

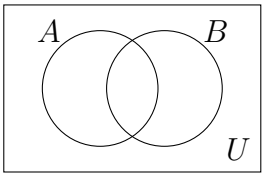


$(A \cup B)^c$

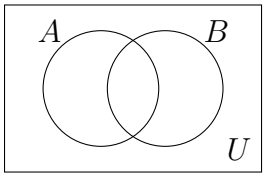


$A^c \cap B^c$

b. Now shade $(A \cap B)^c$ *and* $A^c \cup B^c$.



$(A \cap B)^c$



$A^c \cup B^c$

c. You have just discovered ***De Morgan's laws****. State them as two equations.*

Problem 11 *Draw a Venn diagram of three sets A, B, C in which $A \cap B \neq \varnothing$ and $B \cap C \neq \varnothing$, but $A \cap C = \varnothing$.*

Problem 12

a. A friend claims that $|A \cup B| = |A| + |B|$ for all finite sets A and B. Give a counterexample. For which pairs of sets is the claim actually correct?

b. Prove the correct general formula:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

(Hint: when the right side counts the elements, how many times does each element of $A \cup B$ get counted?)

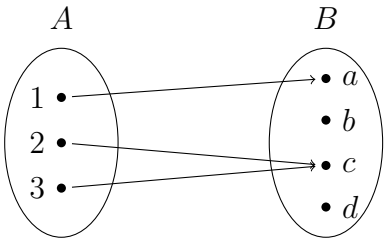
c. Check the formula against the survey of Problem 1: compute the number of students who enjoy chess or programming in two different ways.

Same size, without counting

We can now build new sets at will. The deeper question is how to compare their sizes. Here is the test case for this whole topic: are there more natural numbers $1, 2, 3, \ldots$ than even numbers $2, 4, 6, \ldots$? Write down your instinct; we will settle the matter shortly.

Counting is no help here: counting an infinite set never finishes. Fortunately, there is an older idea, known to every shepherd who cannot count. Such a shepherd drops a pebble into a pouch for every sheep that leaves the pen in the morning, and takes one out for every sheep that returns in the evening. If the pouch is empty when the last sheep walks in, the flock is complete. The shepherd never counted anything: he *matched* sheep with pebbles. Matching is the idea that will replace counting.

A **function** $f\colon A \to B$ is a rule that assigns to each element $a \in A$ exactly one element $f(a) \in B$. Functions can be drawn. Here is one function from $A = \{1, 2, 3\}$ to $B = \{a, b, c, d\}$:



A function $f\colon A \to B$ is called

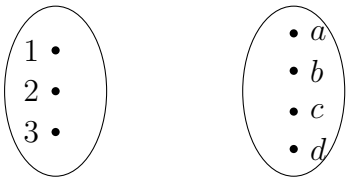
- **injective** if different inputs always give different outputs: $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$;
- **surjective** if every element of B gets hit: $\forall\, b \in B\ \exists\, a \in A$ such that $f(a) = b$ (there are our two symbols);
- **bijective** if it is both.

A bijective function, or a **bijection**, is a perfect matching: every element of A is paired with exactly one element of B, and nothing on either side is left out or used twice.

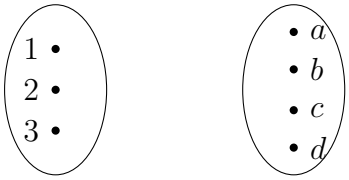
Problem 13 *For each function, decide: injective? surjective?*

- *the function drawn above*
- $f\colon \mathbb{Z} \to \mathbb{Z}, \quad f(n) = n + 3$
- $f\colon \mathbb{Z} \to \mathbb{Z}, \quad f(n) = 2n$
- $f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$

Problem 14 *Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$, as above.*
a. Draw a function $f\colon A \to B$ that is injective, or explain why none exists.



b. Draw a function $f\colon A \to B$ that is surjective, or explain why none exists.



c. In general, what must be true of $|A|$ and $|B|$ for a bijection $f\colon A \to B$ to exist between finite sets?

The last observation is the key. **Definition.** Two sets A and B have the **same cardinality**, written $|A| = |B|$, if there exists a bijection $f\colon A \to B$. For finite sets this agrees with ordinary counting. But note that the definition never mentions counting, so it makes sense for infinite sets too. Time to settle the warm-up question of this topic.

Problem 15 *Let $E = \{2, 4, 6, 8, \dots\}$ be the set of even natural numbers.*
a. Find a bijection $f : \mathbb{N} \to E$, and explain how you know it is injective and surjective.

b. You have just proved $|\mathbb{N}| = |E|$: the even numbers are exactly as numerous as all natural numbers, even though $E \subset \mathbb{N}$. Reconcile this with the instinct you wrote down at the start of the topic.

Problem 16 *Show that $|S| = |\mathbb{N}|$ for the set $S = \{1, 4, 9, 16, 25, \dots\}$ of perfect squares. (Galileo noticed this in 1638 and found it deeply troubling: the squares thin out as you go up the number line, yet the matching is perfect.)*

Problem 17 *Show that $|\mathbb{Z}| = |\mathbb{N}|$: arrange* all *the integers $\dots, -2, -1, 0, 1, 2, \dots$ in a single infinite list with a first entry, a second entry, a third entry, and so on, so that every integer appears exactly once. (A clear pattern is a perfectly good answer. For a bonus, find a formula for your bijection.)*

Problem 18 *Find a function* $f\colon \mathbb{N} \to \mathbb{N}$ *that is injective but* not *surjective. Which natural numbers does your function miss? (Keep your example in mind: it is about to check into a hotel.)*

Hilbert's Hotel

Recall the Grand Hilbert Hotel from Problem 2: infinitely many rooms, numbered $1, 2, 3, \ldots$, and tonight every single one is occupied. A tired traveler is at the desk, and the manager has a promise to keep. The rules of the house: nobody may share a room, nobody may be left without a room, and the manager may broadcast one instruction to all the guests at once (for example: "everyone in room n, please do such-and-such").

Problem 19

a. Find an instruction that frees up a room for the traveler. Then compare with the guess you wrote down in Problem 2.

b. The next night, the hotel full as always, a group of 100 *travelers arrives. What instruction works now?*

Problem 20 *The following night, an* infinite bus *pulls up, carrying infinitely many passengers in seats* $1, 2, 3, \ldots$ *Find an instruction for the guests, and a rule telling each passenger which room to take, so that everyone gets a room of their own. (Hint: Problem 18 had more than one good answer.)*

Problem 21 *Disaster: an* infinite convoy *arrives, buses B_1, B_2, B_3, ..., each carrying infinitely many passengers. Can the manager still house everyone, giving each person a room of their own? Try to find a way, or an argument that it is impossible. (One idea to explore: could products of powers of primes, like $2^i 3^j$, help assign rooms? Check with an instructor before moving on.)*

Counting the fractions

Let us call a set **countable** if it has the same cardinality as $\mathbb{N}$, that is, if its elements can be arranged in a single infinite list in which every element appears exactly once. You have already shown that the even numbers, the perfect squares, and even $\mathbb{Z}$ are countable.

What about the positive fractions? They look far too crowded to list: between any two fractions there is a third one (can you see why?), so there is no "next fraction" after $\frac{1}{2}$, and any attempt to list them in increasing order is doomed. Cantor found a way anyway. So can you.

Problem 22

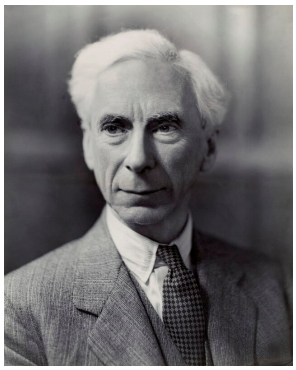
a. Show that the set of all positive fractions is countable: describe a list $r_1, r_2, r_3, \ldots$ *in which every positive fraction shows up after finitely many steps. The list does not have to be increasing. (Hint: this is Problem 21 in disguise.)*

b. If your list is like most first attempts, every positive rational number *appears in it many times, for instance* $\frac{1}{1} = \frac{2}{2} = \frac{3}{3}$*. Explain how to repair the list so that every positive rational number appears exactly once.*

c. Conclude that the set of all *rational numbers, positive, negative, and zero, is countable too.*

When sets go wrong

So far our fearlessness with infinite sets has been rewarded every time. Let us see what happens when we get *too* fearless. Sets can contain other sets; power sets are built of nothing else. It is tempting to go all the way and imagine the grandest set of all: *the set of all sets*. In 1901 Bertrand Russell discovered what happens if you do, and shook the foundations of mathematics.



Bertrand Russell, the man who broke mathematics.
(It has since been repaired.)

Here is his idea, disguised as a story.

Problem 23

a. In a certain village there is exactly one barber. The barber shaves precisely those villagers who do not shave themselves, no more and no less. Who shaves the barber? Show that both *possible answers lead to a contradiction. What is the only way out of the paradox?*

b. Now the real thing. Most sets are not elements of themselves: $\mathbb{N}$ *is not a natural number. Call a set* ordinary *if it is not an element of itself, and consider the set of* all *ordinary sets:*

$$R := \{\, S : S \text{ is a set and } S \notin S \,\}.$$

Is $R \in R$*? Show that both answers lead to a contradiction.*

c. In the story, the way out was that no such barber can exist. What is the way out here? What does it say about "the set of all sets"?

Russell's paradox forced mathematicians to rebuild set theory from the ground up, on **axioms**: precise rules for which collections may be called sets. In the standard system, not every description defines a set, and "the set of all sets" does not exist. The repairs held: all of modern mathematics now rests on this foundation.

One mystery remains. You have shown that $\mathbb{Z}$ and even $\mathbb{Q}$ are countable, so you might guess that every infinite set is. In 1874 Cantor proved otherwise: the set $\mathbb{R}$ of *real* numbers cannot be arranged in any list. No snake path, no hotel trick, no cleverness can exhaust it: some infinities are genuinely bigger than others. (Ask your instructor about the *diagonal argument*; it is a story for another day.) And once there are two sizes of infinity, a question appears between them. You will meet it at the very end of this packet.

Challenge Mix

Here A, B, C are arbitrary sets, living inside an arbitrary universal set. For each statement, decide whether it is necessarily true (T) or not necessarily true (N), and check the box. Hint: do not fall for the trick questions, I believe in you!

1. $\{1, 2\} = \{2, 1\}$ T ☐ N ☐
2. $(1, 2) = (2, 1)$ T ☐ N ☐
3. $\{\varnothing\} \subseteq A$ T ☐ N ☐
4. $A \subseteq B$ implies $A \cup B = B$ T ☐ N ☐
5. if $A \not\subseteq B$, then some $a \in A$ has $a \notin B$ T ☐ N ☐
6. $A \subseteq B$ implies that there exists $b \in B$ such that $b \in A$ T ☐ N ☐
7. $A \subseteq B$ implies $|A| < |B|$ T ☐ N ☐
8. $\mathcal{P}(\varnothing) = \{\varnothing\}$ T ☐ N ☐

9. $A \cap (B \cup C) = (A \cap C) \cup (B \cap C)$ T ☐ N ☐

10. $|A| + |B| = |A \cup B| + |A \cap B|$ T ☐ N ☐

11. at least one holds: $A \cap B \neq \varnothing$, $A \cap B^c \neq \varnothing$, $A = \varnothing$ T ☐ N ☐

Challenge problems

Harder problems are marked with a red pepper sign. Yet harder are marked with two peppers. If you see three peppers, run!

Problem 24 *Set theory can build the numbers themselves out of nothing at all. Define*

$$0 := \varnothing, \quad 1 := \{0\}, \quad 2 := \{0, 1\}, \quad 3 := \{0, 1, 2\}, \quad \ldots$$

and in general $n + 1 := \{0, 1, \ldots, n\}$.

a. Write out the number 3 *in full, using nothing but braces and* $\varnothing$.

b. How many elements does the number n *have?*

c. Show that both $n \in n + 1$ *and* $n \subseteq n + 1$.

Problem 25

a. Find a bijection from the interval $(0, 1)$ *to* all *of* $\mathbb{R}$, *and explain why your function is one. So the whole real line is no bigger than a tiny piece of itself.*

b. Find a bijection from the closed interval $[0, 1]$ *to the open interval* $(0, 1)$. *The two endpoints* 0 *and* 1 *have to be smuggled into the interior, but any honest function must send different points to different points, so there seems to be no room for them... unless you remember how the Hilbert Hotel made room.*

Problem 26 *By Problem 7, a finite set with n elements has $2^n > n$ subsets, so $|\mathcal{P}(A)|$ is strictly bigger than $|A|$.* Cantor's theorem *says this holds for* every *set, finite or not.*

a. Find an injection $A \to \mathcal{P}(A)$, showing the power set is at least as big.

b. Now let $f\colon A \to \mathcal{P}(A)$ be any *function. Each element a is assigned a subset $f(a) \subseteq A$, so it makes sense to ask whether a belongs to its own subset. Let*

$$D := \{\, a \in A : a \notin f(a) \,\}.$$

Show that no element a can satisfy $f(a) = D$. (Split into cases: $a \in D$ or $a \notin D$. Does this remind you of Problem 23?)

c. Conclude that no function $A \to \mathcal{P}(A)$ is surjective, so no bijection exists: $|\mathcal{P}(A)|$ is a strictly bigger size than $|A|$. In particular $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots$: there are infinitely many different sizes of infinity.

Problem 27 *(*The Continuum Hypothesis*; Hilbert's first problem, 1900.) It is known that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$, so by Problem 26 the real line is a strictly bigger infinity than $\mathbb{N}$. Is there a set whose size lies strictly* between *$|\mathbb{N}|$ and $|\mathbb{R}|$? Prove that such a set exists, or prove that it cannot.*

Fine print: Cantor spent years on this question, and Hilbert placed it first on his famous list of problems for the twentieth century. The resolution is stranger than either of them imagined: Gödel (1940) and Cohen (1963) proved that the standard axioms of set theory can never decide it, in either direction. Cohen received the Fields Medal for it. If you solved the problem above, please see an instructor immediately.