THE TRIANGLE INEQUALITY

MATH CIRCLE (INTERMEDIATE) 04/14/2013

Warm-up. Is it possible to have a triangle with the following side lengths? If so, draw one (approximately to scale). If not, explain why not.

(1) 3, 5, and 7?

(a) 3, 5, and 8?

(b) 3, 5, and 9?

The triangle inequality states that for any **nondegenerate**[1] triangle ABC we have three inequalities:

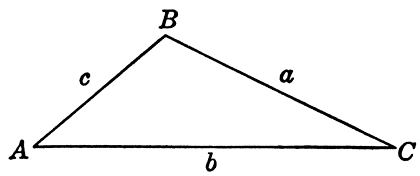

FIGURE 0.1. A triangle.

$$\mathbf{c} < \mathbf{a} + \mathbf{b}$$

$$\mathbf{b} < \mathbf{a} + \mathbf{c}$$
$$\mathbf{a} < \mathbf{b} + \mathbf{c}$$

In other words, **the length of any side of a nondegenerate triangle is less than the sum of the lengths of the other two sides.**

(1) Given $\triangle ABC$ (see Figure 1), with AB =5, $AC = 10$, and $BC = 11$:
   (a) Show that the Triangle Inequality holds.

   (b) Suppose now that $BC = 15$. (and AB and AC are as above). Does triangle ABC exist? Why or why not?

[1]A "**degenerate**" triangle is one in which all three vertices lie on a straight line. From this point forward, if you see "triangle" in a problem, you can assume it is *non*degenerate.

(2) Side AC of $\triangle ABC$ has length 3.8, and side AB has length 0.6. If the length of side BC is an integer, what is its length?

(3) In how many ways can we form a nondegenerate triangle by choosing three distinct numbers from the set $\{1, 2, 3, 4, 5\}$ as the sides?

(4) Prove that for a quadrilateral, the sum of the lengths of any three sides is greater than the length of the fourth side.
(It will definitely help you to begin by drawing an arbitrary quadrilateral.)

(5) Prove that for a hexagon, the sum of the lengths of any five sides is greater than the length of the sixth side.
(It will definitely help you to begin by drawing an arbitrary hexagon.)

Perhaps you are now able to see that this fact will be true for any polygon (and, if you're familiar with mathematical induction, you can prove it at home):
For any polygon, the sum of the lengths of all but one of the sides is greater than the length of the remaining side.

(6) The distance from Leningrad to Moscow is 660 kilometers. From Leningrad to the town of Likovo it is 310 kilometers, from Likovo to Klin it is 200 kilometers, and from Klin to Moscow it is 150 kilometers.
  (a) What can you conclude about the shape (or "shape") formed by the towns Leningrad, Likovo, and Klin?

  (b) How far is it from Likovo to Moscow?

(7) Suppose we have three points on a plane, A, B, and C.

(a) Suppose the distance between points A and B is greater than the distance between points B and C. Draw a picture, and then **prove** that for any three such points A, B, and C we have $AC \geq |AB - BC|$.

(b) Now suppose the distance between points A and B is greater than the distance between points B and C. Can you prove the same result?

(c) What if A, B, and C are collinear (i.e. lie on a line)?

(d) We have now covered all possible situations. Summarize your results.

(8) Prove that the length of any side of a triangle is not more than half its perimeter.

(9) The distance between Town A and Town B is 7 miles. The distance between Town B and Town C is 13 miles. It is known that Towns A, B, and C form a triangle. What is the range of all possible distances between Town A and Town C? (What are the smallest and largest possible distances?)

(10) Prove that the sum of the diagonals of a convex quadrilateral is

(a) less than the perimeter, and

(b) more than half the perimeter.

(11) Prove that the sum of the diagonals of a convex pentagon is

(a) greater than the perimeter, and

(b) less than double the perimeter.

- Review Problems -

(12) Prove the following theorem: In any tree, the number of vertices exceeds the number of edges by 1. That is, $V = E + 1$.

(13) There are 30 towns in a country. Each of them is connected to every other by a single road. What is the maximum number of roads that can be closed in such a way that one can still reach each town from any other?

(14) A volleyball net has the form of a rectangular lattice with dimensions 50x600. What is the maximum number of unit strings you can cut before the net falls apart into more than one piece?
(Hint: you're looking for the maximal tree!)