PROBABILITY I

BEGINNER CIRCLE 4/14/2013

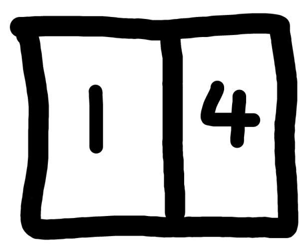
1. Warm-Up: Playing Darts

I hope that you are all familiar with the game of darts. The game goes something like this: A board is set up on the opposite side of the room, with different regions corresponding to different amounts of points. Darts are thrown across the room, and the number of points that you earn is equal to the number on the region the dart lands in.

The math instructors want to play darts. As they are all mathematicians, they have horrible vision, and thus do not play darts very well. The best that they can do is throw darts in such a way that they know that will hit the dart board, but they have no idea *where* on the dartboard the dart will strike. Furthermore, they have a pretty hard time making out where the dart landed when it strikes the other side of the wall.

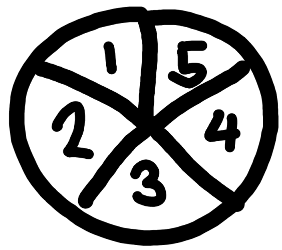
Fortunately, our protagonists are not very picky, and do not really care what the exact score of the game was, but rather, are ok making guesses about the score that they get. For each of the following games, give a rough estimate of the score at the end of shooting darts.

Problem 1. Suppose Isaac throws a hundred darts at this dartboard. What is a good guess for his score?



Jeff Hicks/Los Angeles Math Circle/UCLA Department of Mathematics .

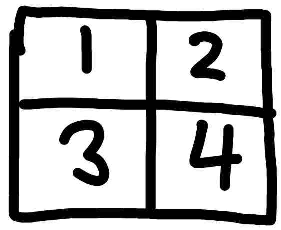
Problem 2. Suppose Derek throws 10 darts at this dartboard. What is a good guess for his score.



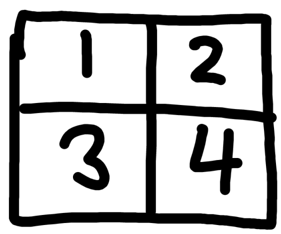
Problem 3. Morgan is a lefty, so 2/3 of his darts go to the dartboard on the left, while the remaining 1/3 go to the dartboard on the right. If he throws 60 darts, what is a good guess for his score?

Problem 4. Jeff plays "misére" darts, which is to say that he cheats. Whenever Jeff throws a dart, he gives himself the score of all the other regions that he missed. If he

throws 20 darts, what is a good guess for his final score?

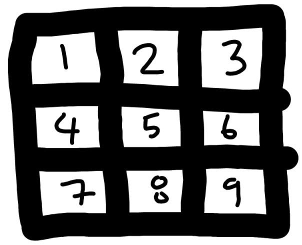


Problem 5. After years of training in a monastery in Japan, Jonathan has perfected the art of "nihon bo shuriken", which allows him to throw two darts at the same time. When he throws the two darts, the score that he gets is the *product* of the scores in the two regions he hits. If he throws $2 \times 40 = 80$ darts, what is a good guess for his score?



Problem 6. Isaac is playing hardcore darts. In hardcore darts, you throw two darts per turn. You get the points only if you throw both darts into the same region. Suppose Isaac throws 2×90 darts at this dartboard. What is a good guess for his score at

the end of the game.



Problem 7. Jeff and Derek begin designing a new dartboard. Because their markers ran out of ink, the best they can do is cut out a large circle, 1 meter in radius. They decide that the number of points that each dart is worth is equal to the distance the dart is away from the edge– $(1 -$ the distance from the center$)$. If they throw 20 darts at this dartboard, what is a good guess for their score at the end of the game?

2. Things that are not darts

How do we describe probability? Probability somehow measures the likelihood that a specific event occurs out of a whole bunch of different events. For example, if we flip a coin, we have 2 different possible outcomes:

$$S = \{\text{heads}, \text{tails}\}$$

The probability that we flip a head is the likelihood that we pick heads out of that set.

Definition 1. The set of outcomes of some probability problem is written with the letter S. For every possible outcome x, we can assign a number telling us how likely that event is to occur, called the probability of x, and written $P(x)$

The function $P(x)$ takes an outcome x and assigns a probability to them. For instance,

$$P(\text{heads}) = \frac{1}{2}$$

means the likelihood of flipping a head is 1 in 2.
The probability function follows two special rules:

(i) Let a and b be two separate outcomes in S. Then the probability of picking outcome a is $P(a)$, while the probability of picking outcome b is $P(b)$. The probability of picking either outcome a or b is

$$P(a \text{ or } b) = P(a) + P(b)$$

(ii) Let $S = \{s_1, s_2, \ldots, s_n\}$ be the possible events of a probability problem. Then the total probability of picking an outcome from S is

$$P(s_1 \text{ or } s_2 \text{ or } \ldots \text{ or } s_n) = 1$$

This is like saying, when you flip a coin, you have a probability of $\frac{1}{1}$ of getting either heads or tails.

Problem 8.

(i) What is the set of outcomes S for flipping a coin?

(ii) If the coin is fair, then the chances of getting heads or tails is the same. Can you write this down using $P(\text{heads})$ and $P(\text{tails})$?

(iii) Since the only possible outcomes are heads and tails, what does $P(\text{heads}) + P(\text{tails})$ equal?

(iv) Using the two earlier sections, use some algebra to show $P(\text{heads}) = \frac{1}{2}$

Problem 9.

(i) When a die is rolled, what is the set S of possible outcomes showing on the die?

(ii) If the die is fair, then the chances of rolling different numbers is the same. Can you write this down using $P(1), P(2), \ldots P(6)$, the probabilities that you roll a $1, 2, 3, \ldots 6$?

(iii) Since the only possible outcomes are $1, 2, 3 \ldots$ or 6, what does that tell us

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) =?$$

(iv) Using the two earlier sections, conclude that $P(1) = \frac{1}{6}$

Notice that the above problems rely on the fact that you know that the outcomes are equally likely. However, sometimes the outcomes are not equally likely!

Problem 10. If you flip two coins, and you can tell them apart, then there are four different possible outcomes,

$$S = \{HH, HT, TH, TT\}$$

(i) What is the probability that you get two heads? Explain your solution using full sentences.

(ii) What is the probability that you get a head and a tail (this can happen two different ways!). Explain your solution in full sentences

(iii) What is the probability that you do not get 2 heads? Explain your solution in full sentences.

Problem 11.

(i) Two dice are rolled. How many different outcomes are there? (note: if you roll a 5 and a 6, it is different than rolling a 6 and 5!)

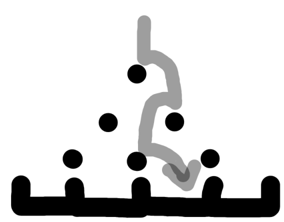
(ii) How many ways can you can you roll a 2?

(iii) How many ways can you roll a 3?

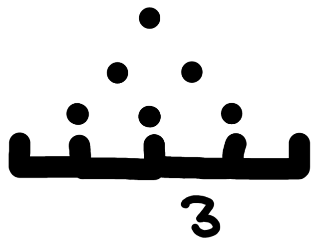
(iv) Explain, in full sentences, why $2 \times P(2) = P(3)$? (The probability of rolling a 3 is twice as much as rolling a 2).

(v) What is the probability of rolling a 3?

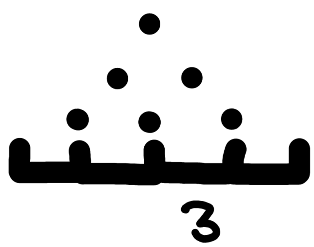
Problem 12. A pachinko machine is set up, and balls bounce from the top of the machine to the bottom. Suppose the probability of the ball taking any path from the top to the bottom is the same. What is the probability of the ball falling along the given path? (Hint: How many paths are there from the top to the bottom)



Problem 13. In the pachinko machine above, what is the probability that the ball falls into the bin labeled 3? How did you arrive at your solution?



Problem 14. What is the probability that the ball does not fall into the bin labeled 3? How did you arrive at your solution?



Problem 15. Suppose that we roll a 6 sided die. Then the number of outcomes that die can roll is 6. What is the number of outcomes of rolling a blue die and a red die? What is the number of outcomes rolling a red and a blue and a green die?

Problem 16. If you roll 2 die, what is the probability that you roll a 8?

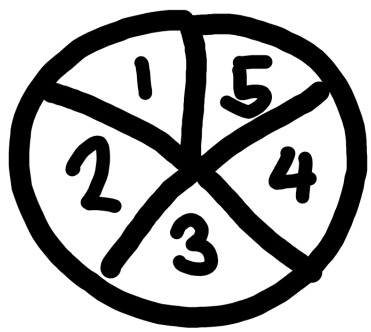
Problem 17. If you roll 3 die, what is the probability that you roll a 8?

3. Making dartboards out of things that should not be dartboards

We can convert a probability problem into a dartboard problem and vice versa. Let S be a set of outcomes, and P a probability function that gives probabilities to each outcome in S.
On the flipped, we have a set of scores, and a dartboard with different regions on it. The link between these two is

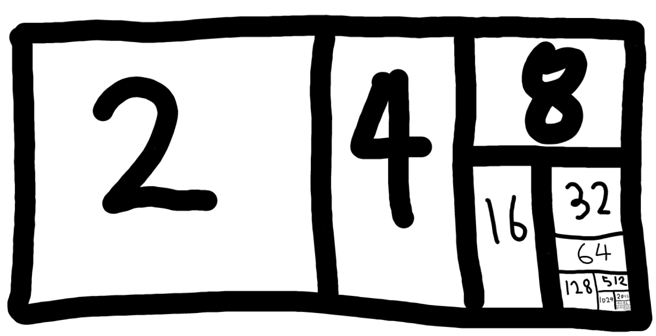
The set of outcomes, S	$\Leftrightarrow$	The set of scores
P the probility function	$\Leftrightarrow$	The area of each region

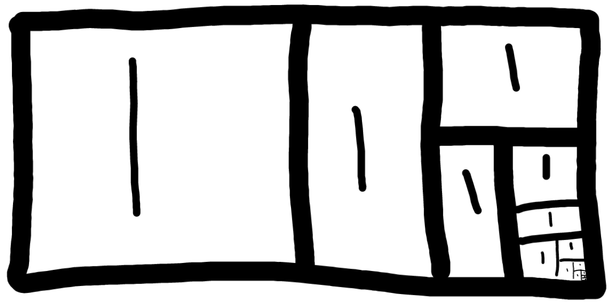
Problem 18. What is the probability associated to each region on this dartboard?



Problem 19. Draw a dartboard whose scores give the probability of rolling die.

Problem 20. What is the average score when you throw a dart at these funny dartboards here? (Hint: convert it into a probability problem)





Problem 21. If we have two different dartboards, we can make a “mesh dartboard”, but crossing them together. The probability of shooting it into region 1, and then into region 2 is given by the crossed regions. What is the probability of shooting it into the 1, and then the 2?