UCLA Math Circle 4/14/2013

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Last Time: Pick's Theorem

Theorem 0.1 (Pick's Theorem) *Let P be a polygon with its vertices at lattice points. If $A(P)$ is the area of P, $i(P)$ is the number of lattice points in the interior of P and $b(P)$ is the number of lattice points on the boundary of P, then*

$$A(P) = i(P) + \frac{b(P)}{2} - 1.$$

Theorem 0.2 *Every polygon (polytope) has a triangulation.*

Remark Is there a Pick's Theorem for higher dimensions? Can volume be expressed in terms of the number of lattice points on the surface of a polytope and the number of lattice points in its interior?

Today: Ehrhart Theory

Helpful Formulas

- $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.
- $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.
- $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

Lecture Content

Definition Let P be a polytope in $\mathbb{R}^n$ (i.e. n-dimensional space), and suppose all the coordinates of all the vertices of P have integer coordinates. For every positive integer t define $f_P(t)$ to be the number of points with integer coordinates in the interior or on the boundary of tP, the t-th dilation of P. Remember, tP is

$$tP := \{(tx_1, tx_2, \ldots, tx_n) \ : \ t \in P\}$$

Theorem 0.3 (Ehrhart's Theorem) *If P is a polytope whose vertices all have integer coordinates, then $f_P(t)$ is a polynomial in t with degree equal to the dimension of P.*

Theorem 0.4 *If P is a polytope whose vertices all have integer coordinates, and P has dimension d, then the t^d coefficient of $f_P(t)$ is equal to*

Theorem 0.5 *If P is a polytope whose vertices all have integer coordinates, and P has dimension d, then the constant coefficient of $f_P(t)$ is equal to*

Theorem 0.6 *If P is a polytope whose vertices all have integer coordinates, and P has dimension d, then the t^{d-1} coefficient of $f_P(t)$ is equal to*

Theorem 0.7 *If P is a polytope whose vertices all have integer coordinates, and P has dimension d, then $f_P(-1)$ is equal to*