Quadratic Forms

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What does each of the following curves in the plane look like?

1. $x^2 + y^2 - 1 = 0$
2. $x^2 = 1$
3. $x^2 - y^2 = 1$
4. $xy = -1$
5. $x^2 + 2xy - y^2 = 0$
6. $x^2 + xy + y^2 = 1$
7. $x^2 + y^2 + 2xy - 2x - 2y - 3 = 0$
8. $3x^2 + 3y^2 - 2xy - 10x + 6y + 5 = 0$
9. $4xy + 2x - 6y - 3 = 0$

What does each surface in space look like?

1. $x^2 + y^2 + z^2 = 0$
2. $x^2 + y^2 - z^2 = 1$
3. $11x^2 + 2y^2 + 2z^2 - 10xy - 10xz + 4yz - 4x + 2y + 2z - 8 = 0$

Quadratic Forms

When trying to understand conic sections and quadric surfaces, it is advantageous to start by just looking at the second order terms. A *quadratic form* in x, y is a function of the form $Q(x,y) = Ax^2 + Bxy + Cy^2$. In three variables it takes the form $Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2$.

1. For each of the following quadratic forms $Q(x,y,z)$, find a change of variables $(x,y,z) \mapsto (u,v,w)$ such that $Q(u,v,w) = \pm u^2 \pm v^2 \pm w^2$.
   (a) xy
   Solution: Let $u = \frac{1}{2}(x+y)$, $v = \frac{1}{2}(x-y)$. Then $xy = u^2 - v^2$
   (b) $x^2 + xy + y^2$
   (c) $5x^2 + 2y^2 + 2z^2 - 4xy - 2yz$
   (d) $x^2 + y^2 + z^2 + 6xy + 2xz - 2yz$
   (e) $x^2 + y^2 + 4xy + 2xz + 2yz$

Bilinear Forms

Sometimes it's useful to study a more general type of object than the one we might be interested in. A *bilinear form* in the plane is a function which takes in *two* vectors $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x' \\ y' \end{bmatrix}$, and outputs a value

$$B(\vec{u}, \vec{v}) = Axx' + Bxy' + Cx'y + Dyy'.$$

Sometimes, if $B(\vec{u}, \vec{v})$ is a bilinear form, $Q(\vec{u}) = B(\vec{u}, \vec{u})$ defines a quadratic form as above, but not always (see Exercise 2).

1. Does $B((x,y),(x',y')) = xy' - x'y$ gives rise to a quadratic form? Which one? Do any other bilinear forms give rise to the same quadratic form?

2. (*Does every quadratic form come from a bilinear form?*) Show that if a bilinear form $B(\vec{u}, \vec{v})$ gives rise to a quadratic form $Q(\vec{u})$ then Q must satisfy the *polarization identity*

$$B(\vec{u}, \vec{v}) = \frac{1}{2}(Q(\vec{u} + \vec{v}) - Q(\vec{u}) - Q(\vec{v})).$$

   In fact, if Q is any function satisfying $Q(\alpha\vec{u}) = |\alpha|^2 Q(\vec{u})$, and such that the right hand side of this equation is a bilinear form, then Q must be a quadratic form.

3. (Complex Polarization) Assume that Q is a *complex quadratic form*, meaning that the input to a quadratic form can be scaled by a complex number, yielding an the output which is in general a complex number, and satisfying the same hypothesis as before. Find a formula for the bilinear form $B(\vec{u}, \vec{v})$ which gives rrise to $Q(\vec{u})$.