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Calculus Packet

Calculus studies two important ideas:

1. How things change
2. How things accumulate

To understand change, we begin with slopes. Recall from algebra that slope measures how steep a line is:

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}.$$

For curved graphs, however, the slope changes from point to point.To estimate slope on a curve, we first use **secant lines**, lines connecting two nearby points on the graph.

As the second point moves closer and closer, the secant line approaches a **tangent line**. The slope of the tangent line is called the **derivative**.

Problem 1 *For what type of function does the secant line between any two points always have the same slope as the tangent line at every point?*

Part I: Slopes and Tangent Lines

Consider the graph of the function

$$y = x^2.$$

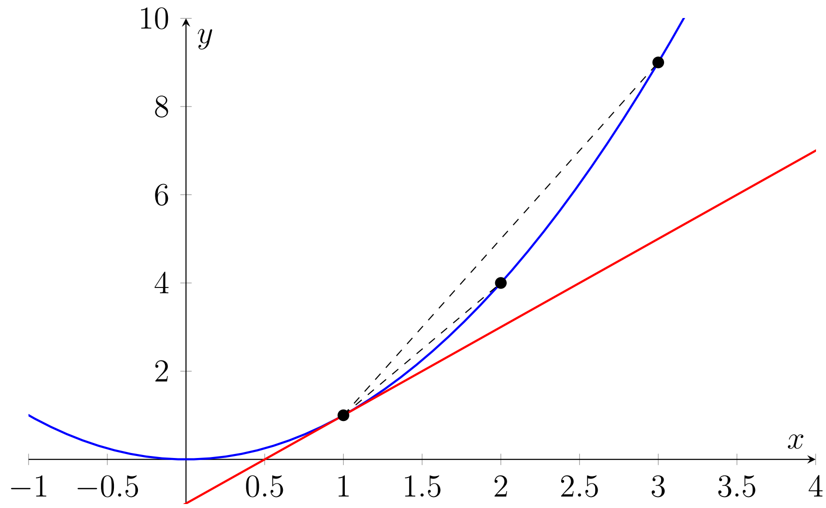


Figure 1: A parabola with secant and tangent lines

Problem 2 *Answer the following:*

1. *Which line above appears to best approximate the tangent line?*
2. *Estimate the slope of the tangent line at* $(1, 1)$*.*
3. *What happens as the second point moves closer to* $(1, 1)$*?*

Average rate of change measures how much a function changes over an interval.

For example, for the function

$$f(x) = x^2,$$

the average rate of change between $x = a$ and $x = b$ is

$$\frac{f(b) - f(a)}{b - a}.$$

Problem 3 *Answer the following:*

1. *Find the average rate of change of* $f(x) = x^2$ *from* $x = 1$ *to* $x = 3$.
2. *Find the average rate of change from* $x = 1$ *to* $x = 2$.
3. *Find the average rate of change from* $x = 1$ *to* $x = 1.1$.
4. *What value do these slopes appear to approach?*

Suppose the position of a particle is given by

$$s(t) = t^2 + 2t.$$

Average velocity is computed using

$$\frac{s(b) - s(a)}{b - a}.$$

Problem 4 *Answer the following:*

1. *Find the average velocity on the interval* $[0, 1]$.
2. *Find the average velocity on the interval* $[1, 2]$.
3. *Find the average velocity on the interval* $[2, 2.1]$.
4. *Estimate the instantaneous velocity at* $t = 2$.

Part II: Derivatives

A derivative measures instantaneous rate of change. For a function $y = f(x)$, the derivative is written

$$\frac{dy}{dx}.$$

Geometrically, the derivative gives the slope of the tangent line.

Constant Rule:

$$\frac{d}{dx}(c) = 0$$

Constant Multiple Rule:

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Sum Rule:

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

Difference Rule:

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Chain Rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Problem 5 *Answer the following:*

1. *Find the derivative of* $f(x) = x$.
2. *Find the derivative of* $f(x) = 5x^2$.
3. *Find the derivative of* $f(x) = 6x^2 + 3x + 6$.

Problem 6 *Answer the following:*

1. *Find the derivative of* x^3.
2. *Find the derivative of* x^4.
3. *Find the derivative of* x^5.
4. *What pattern do you notice?*

Problem 7 *Answer the following:*

1. *Find the equation of the tangent line to*

$$y = x^2$$

at $x = 2$.

2. *Find all points where*

$$f(x) = x^3 - 3x$$

has horizontal tangent lines.

Part III: Integration

Integration studies accumulation and area, but it is also helpful to think of integration as the operation opposite to differentiation. Let D be the differentiation operator. This means that D takes a function and gives its derivative:

$$D(f(x)) = f'(x).$$

For example,

$$D(x^3) = 3x^2.$$

Now let S be the integration operator. For a function $f(x)$, define

$$S(f(x)) = \int_a^x f(t)\,dt.$$

Here, $S(f(x))$ means we are accumulating the values of $f(t)$ from $t = a$ to $t = x$. The important idea is that differentiation and integration undo each other:

$$D(S(f(x))) = f(x).$$

Also, in a less formal way,

$$S(D(f(x))) = f(x).$$

We are not going to prove these facts in this packet. Instead, we will use them to understand where the rules of integration come from.

For example, since

$$\frac{d}{dx}(x^3) = 3x^2,$$

we know that

$$\int 3x^2 \, dx = x^3 + C.$$

So integration asks the reverse question:

What function had this derivative?

Problem 8 *Answer the following:*

1. *Since*

   $$\frac{d}{dx}(x^2) = 2x,$$

   what is

   $$\int 2x \, dx?$$

2. *Since*

   $$\frac{d}{dx}(x^4) = 4x^3,$$

   what is

   $$\int 4x^3 \, dx?$$

3. *Since*

   $$\frac{d}{dx}(5x) = 5,$$

   what is

   $$\int 5 \, dx?$$

Now we derive the power rule for integration from the power rule for differentiation. Recall that

$$\frac{d}{dx}(x^{n+1}) = (n+1)x^n.$$

Therefore, if we divide by $n+1$, we get

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n.$$

So,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1.$$

The $+C$ appears because constants disappear when we differentiate. For example,

$$\frac{d}{dx}(x^2) = 2x,$$

but also

$$\frac{d}{dx}(x^2+7) = 2x.$$

So both x^2 and x^2+7 have the same derivative. This is why we write

$$\int 2x\,dx = x^2 + C.$$

Problem 9 *Use differentiation to check each integration formula:*

1.

$$\int x^2\,dx = \frac{x^3}{3} + C$$

2.

$$\int x^5\,dx = \frac{x^6}{6} + C$$

3.

$$\int 7x^6\,dx = x^7 + C$$

Problem 10 *Derive the following integration rules from differentiation rules:*

1. *Use*

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

to explain why

$$\int cf(x)\,dx = c\int f(x)\,dx.$$

2. *Use*

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

to explain why

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

3. *Use*

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

to explain why

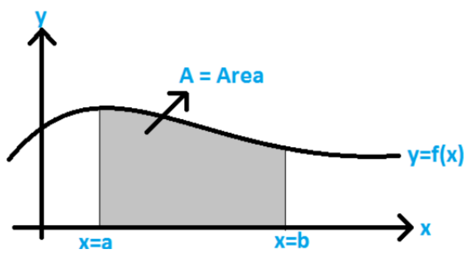
$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx.$$

For a function $y = f(x)$, the integral is written

$$\int f(x)\,dx.$$

Integrals often represent area under a curve.

For a better visualization, here is a picture of integration at work.



Here are some useful integration rules.

Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

$$\int x^{-1}\,dx = \ln|x| + C$$

Constant Multiples

$$\int kf(x)\,dx = k\int f(x)\,dx$$

Sums and Differences

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

Exponential

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

Absolute Value

$$\int |x|\,dx = \frac{x|x|}{2} + C$$

Part IV: Fundamental Theorem of Calculus

The rule connecting integrals and derivatives is known as the Fundamental Theorem of Calculus. Derivatives and integrals are inverse operations.

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$$

With this in mind, let us consider the function $x(t)$, where t is time, and x is distance from the origin. To know the rate at which the object moves relative to the origin, we take the derivative of position. To know the rate of change of velocity, we take another derivative.

$$\begin{aligned} \text{Position: } & x(t) \\ \text{Velocity: } & v(t) = \frac{dx}{dt} \\ \text{Acceleration: } & a(t) = \frac{dv}{dt} \end{aligned}$$

Position is obtained by integrating acceleration twice. Assume no air resistance.

Acceleration:

$$a_x(t) = 0, \qquad a_y(t) = -g$$

Initial conditions at time $t = 0$:

$$x(0) = x_0$$

$$y(0) = y_0$$

$$v_x(0) = v_{0x}$$

$$v_y(0) = v_{0y}$$

Velocity (integrating acceleration):

$$v_x(t) = v_{0x}$$

$$v_y(t) = v_{0y} - gt$$

Position (integrating velocity):

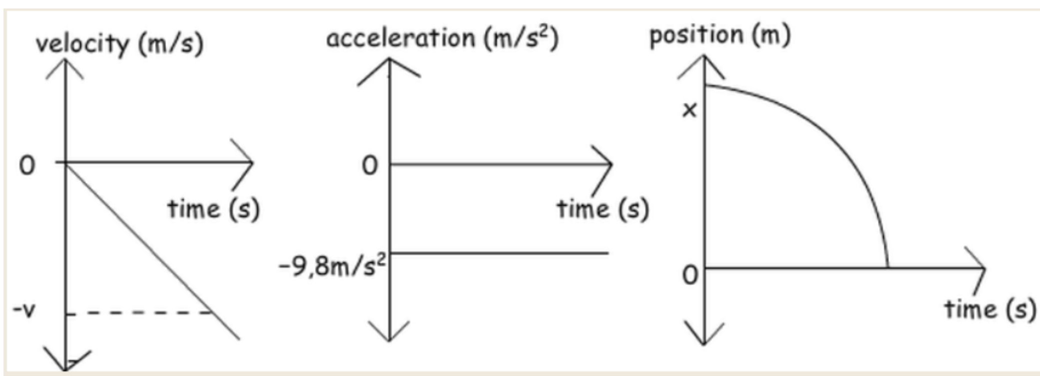
$$x(t) = x_0 + v_{0x}t$$

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

If we let

$$g = 9.8m/s^2,$$

we can graph the following.



Problem 11 *Answer the following:*

1. *Suppose the acceleration of a particle is given by*

$$a(t) = 6t.$$

Assume that

$$v(0) = 0, \qquad x(0) = 0.$$

   (a) *Find the velocity function $v(t)$.*
   (b) *Find the position function $x(t)$.*

Problem 12 *Answer the following:*

1. *Suppose the acceleration of a particle is given by*

$$a(t) = 4.$$

Assume that

$$v(0) = 2, \qquad x(0) = 1.$$

   (a) *Find the velocity function* $v(t)$.

   (b) *Find the position function* $x(t)$.

Problem 13 *Answer the following:*

1. *Suppose the acceleration of a particle is given by*

$$a(t) = 3t.$$

Assume that

$$v(0) = 1, \qquad x(0) = 0.$$

   (a) *Find the velocity function* $v(t)$.

   (b) *Find the position function* $x(t)$.

Problem 14 *Answer the following:*

1. *Suppose the acceleration of a particle is given by*

$$a(t) = 2t + 1.$$

Assume that

$$v(0) = 0, \qquad x(0) = -2.$$

   (a) *Find the velocity function* $v(t)$.

   (b) *Find the position function* $x(t)$.

Problem 15 *Answer the following:*

1. *Suppose the acceleration of a particle is given by*

$$a(t) = 6t^2.$$

Assume that

$$v(1) = 3, \qquad x(1) = 2.$$

   (a) *Find the velocity function* $v(t)$.

   (b) *Find the position function* $x(t)$.

Problem 16 *Answer the following:*

1. *A rocket burns fuel at the rate*

$$r(t) = 500 - 20t$$

   kilograms per second.

   (a) *Find the total amount of fuel used during the first* 10 *seconds.*

   (b) *Explain why integration gives the total fuel consumed.*

Problem 17 *Answer the following:*

1. *The velocity of a spacecraft is given by*

$$v(t) = 3t^2 + 2t$$

   meters per second.

   Assume:

$$x(0) = 0.$$

   (a) *Find the position function* $x(t)$.

   (b) *Find the distance traveled after* 4 *seconds.*

Problem 18 *Answer the following:*

1. *NASA is testing a small research rocket.*
   Assume the rocket moves vertically upward.
   The acceleration of the rocket is given by

   $$a(t) = -9.8 \ m/s^2.$$

   Suppose the rocket is launched from ground level with initial velocity

   $$v(0) = 120 \ m/s,$$

   and initial position

   $$y(0) = 0.$$

   (a) *Find the velocity function* $v(t)$.
   (b) *Find the position function* $y(t)$.
   (c) *Determine when the rocket reaches its maximum height.*
   (d) *Determine the maximum height reached by the rocket.*
   (e) *Determine when the rocket returns to the ground.*

Problem 19 *Answer the following:*

1. *A cannonball is fired from ground level with an initial horizontal velocity of*

$$v_x(0) = 40 \ m/s$$

and an initial vertical velocity of

$$v_y(0) = 60 \ m/s.$$

Assume the acceleration due to gravity is

$$a_x(t) = 0, \qquad a_y(t) = -9.8 \ m/s^2.$$

Assume the cannon is fired from the origin:

$$x(0) = 0, \qquad y(0) = 0.$$

   (a) *Find the horizontal velocity function* $v_x(t)$.

   (b) *Find the vertical velocity function* $v_y(t)$.

   (c) *Find the horizontal position function* $x(t)$.

   (d) *Find the vertical position function* $y(t)$.

   (e) *Find the maximum height reached by the cannonball.*

   (f) *Determine when the cannonball hits the ground.*

   (g) *Determine how far the cannonball travels horizontally before hitting the ground.*